

ELEMENTARY PLANE GEOMETRY

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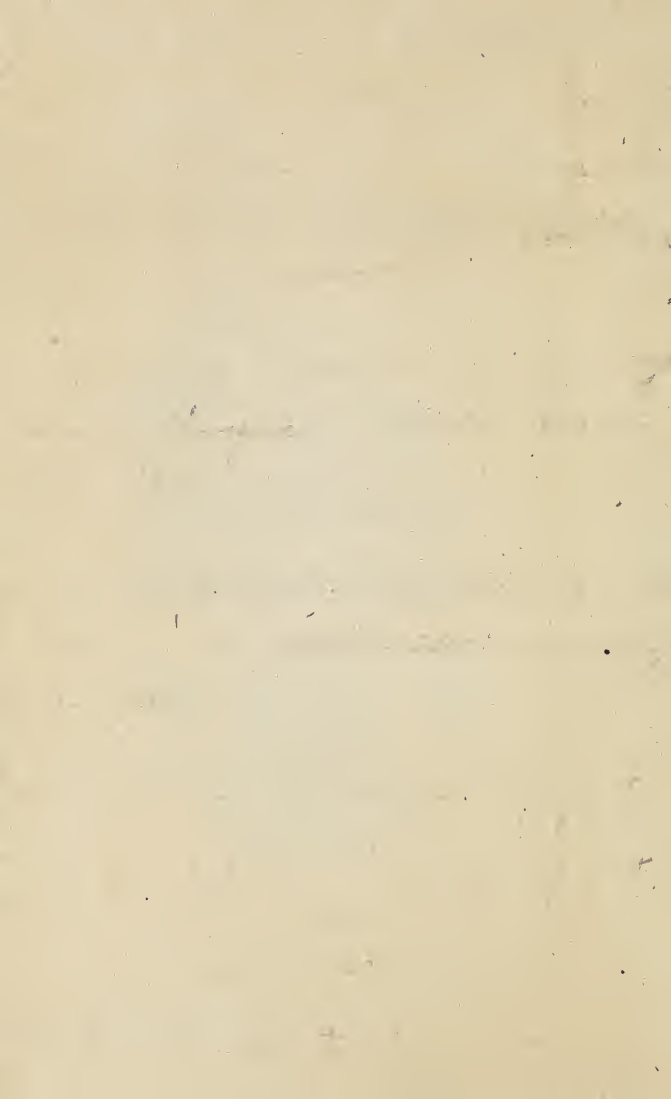
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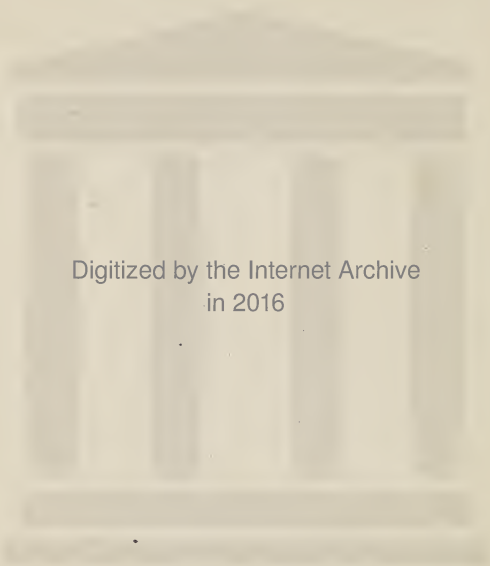
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ELEMENTARY PLANE GEOMETRY

INDUCTIVE AND DEDUCTIVE

BY

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WESTERN EDITION

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EDITORS' PREFACE

THE present edition of this Elementary Plane Geometry is not in any sense intended as a revision of Professor Baker's work. It is rather an attempt on our part to adapt the previous edition, which was used largely as a High School text, to the work of entrance classes. To this end we have endeavoured to prepare a book which will be of use, as well to the large proportion of pupils who leave school after Grade VIII, as to those who may expect to engage in further study upon this subject in High School.

The following are some of the points of difference between this edition and the original text :

- (1) Formal proofs of theorems by the Euclidean method have been practically eliminated.
- (2) Deductive reasoning has been used only in its simplest forms, and where it may serve to impress upon the mind of the pupil the truths previously discovered by experiment.
- (3) Special emphasis has been laid on practical applications of geometrical truths. An appendix containing a set of miscellaneous practical examples has been added.
- (4) The metric system of measurement has not been used in this edition, it being considered necessary to acquaint our pupils more thoroughly with our own units of measurement.

- (5) Many of the more difficult exercises at the end of each chapter have been simplified or omitted entirely. Those remaining have been carefully selected and graded. A number of new exercises have also been added.
- (6) Mainly for the guidance of teachers, a summary has been placed at the end of each chapter in which are set forth all the important points dealt with in that chapter.

Thus, while we have endeavoured to preserve the spirit and characteristic features of Professor Baker's work, we trust that such alterations and additions as we have made will make this book more valuable as a text for use in the public schools.

Our thanks are specially due to Professor Baker who has left us perfectly free to make such alterations as we should deem necessary to meet the conditions of our work.

J. C. PINCOCK
A. C. CAMPBELL

WINNIPEG, March 31, 1914

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INSTRUMENTS.

In the pages that succeed, the following instruments are essential :

1. **A ruler or straight-edge,**

on which are marked inches divided into sixteenths, and on which also is a scale giving millimetres.

This is used for drawing straight lines ; for making them of any required length ; and for measuring straight lines that are drawn.

2. **A pair of compasses,**

one leg of which is furnished with a pencil.

This is used for describing circles ; also; with the help of the ruler, for laying off required distances ; and for measuring distances that are laid off.

3. **A protractor.**

This is used for constructing angles of any given number of degrees ; and for measuring the number of degrees in any given angle. It may also be used for determining whether one angle is greater than, equal to, or less than another.

For the more rapid and more accurate construction of figures, the following instruments are also desirable :

4. **A pair of dividers,**

both the legs of which terminate in fine points. These more accurately than the compasses will enable the pupil to measure and to transfer distances.

5. **A set-square.**

The right angle has very frequently to be constructed, and its construction can be more rapidly effected with the set-square than with the protractor.

6. A bevel.

This enables us very rapidly to determine the equality or inequality of angles, and to construct an angle equal to another.

7. Parallel rulers.

While for drawing lines parallel to each other nothing more is essential than a ruler along which the set-square is made to slide, or a ruler and an instrument for measuring angles, or a ruler and compasses, these methods become tedious from the frequency with which the construction has to be made. Parallel rulers make the construction rapidly and accurately.

Care should be taken to use a pencil with a hard fine point, so that lines drawn may be narrow and well defined.

Smooth paper will be found better than rough.

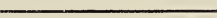
Points and the ends of lines should be marked by indentations made with a needle or with the sharp points of the dividers.

A piece of smooth, perfectly flat board, about a foot square, will be found useful as a drawing board.

In all cases the pupil should construct for himself the necessary figures, and not content himself with those in the book, which are merely intended as suggestions. It will be usually found desirable to make figures on a larger scale than those in the text.

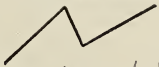
CHAPTER I.

Geometrical Elements.

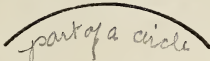
A Straight Line: 

It is evidently the shortest distance between its ends.

A broken line:

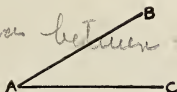

a line with more than 1 direction.

A curved line:

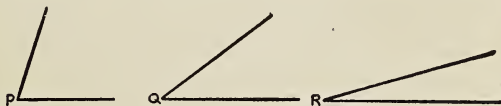

part of a circle

An angle:

space between 2 lines.

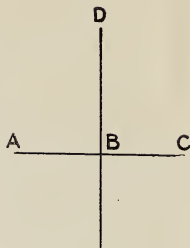


The size of the angle does not depend on the lengths of the bounding lines **AB** and **AC**, but on the *amount of divergence* of these lines from one another. Thus the angle **P** is greater than the angle **Q**, and the angle **R** is less than the angle **Q**.



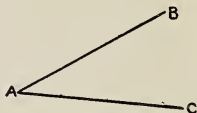
It is usual to indicate an angle by using one letter, as the angle **P**, or by using three letters, as the angle **BAC**. In the latter case the letter at the angle itself is in the middle, and the other two letters lie on the *arms* of the angle.

If the line **DB** cuts the line **AC** at the point **B**, so as to make the angle **DBA** equal to the angle **DBC**, then each of these angles is called a **right angle**, and the lines **DB** and **AC** are said to be perpendicular to each other.

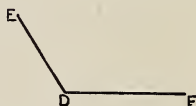


If the two angles on the other side of the line be measured, they will be found to be equal, and hence are also right angles.

2 An angle which is less than a right angle, as **BAC**, is called an **acute angle**.



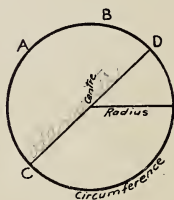
3. An angle which is greater than a right angle, as **EDF**, is called an **obtuse angle**.



A **circle** is the usual figure described on a flat surface by means of the compasses.

Note the parts called **centre**, **radius**, and **circumference**.

All radii of the same circle are equal, since the ends of the compass legs remain the same distance apart while the circle is being described.



A line through the centre and terminated both ways by the circumference is called a **diameter**, as **CD**.

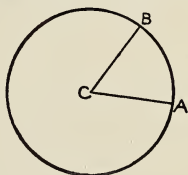
The part of the circle on each side of a diameter is called a **semicircle**.

A part of the circumference, as **AB**, is called an **arc** of the circle. The straight line joining **A** and **B** is called a **chord**.

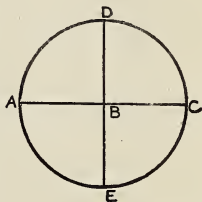
Any line drawn from a point without the circle and cutting it, is called a **secant**.

The circumference of any circle is supposed to be divided into 360 equal parts, each part being called a **degree**.

If the arc **AB** contains 60 degrees, then the angle **ACB** at the centre is an angle of 60 degrees, expressed by 60° .



The lines **AC**, **DE**, through the centre, being perpendicular, each of the arcs **AD**, **DC**, **CE**, **EA** must contain 90° , and the angles **ABD**, **DBC** . . . are angles of 90° .



A semicircle contains 180° , and the straight angle **ABC** contains 180° .

A **triangle**:

It has three sides and three angles.



A **quadrangle**:

It has four angles. Having four sides, it is also called a **quadrilateral**.



A straight line joining two opposite corners of a quadrilateral is called a **diagonal**.

Figures contained by more than four straight lines are called **polygons**.

A straight line has evidently throughout its entire length the same *direction*.

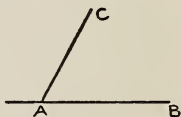
Two straight lines which have the same direction are said to be **parallel** to one another.



Parallel straight lines cannot intersect. For if they did, at the point of intersection they would have different directions, and would therefore have different directions throughout their entire lengths, and hence would not be parallel.



To construct with the protractor at the point **A** in the line **AB** an angle of any required magnitude, say 63° : Place the centre of the protractor at **A**, and let the line joining the centre with the point on the circumference which indicates 0° , rest along **AB**. At the point where the 63° line meets the circumference make a fine mark, **C**, on the paper. Removing the protractor, join **AC**. The angle **BAC** is of magnitude 63° .



Exercises.

All figures in this and succeeding exercises must be accurately constructed with instruments.

1. With the compasses describe circles having radii of lengths 5, 7, 10, . . . sixteenths of an inch.

2. With the protractor construct angles of magnitude 10° , 15° , 25° , 30° , 37° , 43° , ...

3. With the bevel construct a second set of angles of the foregoing magnitudes, using these angles to set the bevel.

4. Draw five straight lines of different lengths, and with the dividers and rule measure their lengths in inches and sixteenths of an inch.

5. Construct five angles, and, using the bevel, determine which is greatest and which least. Arrange them in order of magnitude. Using the protractor, measure their magnitude to the nearest degree.

6. Draw five straight lines of different lengths, and with the eye endeavor to judge their lengths in inches and fractions of an inch. Afterwards test the correctness of your judgment by actually measuring the lines.

7. Construct five angles of different magnitudes, and with the eye endeavor to judge the number of degrees in each. Afterwards test the correctness of your judgment by actually measuring the angles with the protractor.

8. With the eye endeavor to judge the lengths or heights of various objects in the room, at a distance from you. Afterwards test the correctness of your judgment by actually measuring the lengths or heights.

9. A and B being two distant objects and your eye being at C, endeavor with the eye to judge the angle which these objects subtend at your eye, *i.e.*, the angle ACB. Afterwards sight the inside edges of the legs of the bevel towards A and B, and then placing the bevel on the protractor, roughly measure in this way the angle ACB, so correcting, if necessary, your judgment.

10. Draw any two lines of different lengths, and draw a line equal to their difference.

11. Draw any line, and draw another line three times as long as the former.

12. Construct two angles of different magnitudes, and with the bevel constructing two adjacent angles equal to them, form an angle equal to their difference. Measure with the protractor the number of degrees in the original angles and in the difference, and compare.

13. Construct two angles of different magnitudes, and with the bevel constructing two adjacent angles equal to them, form an angle equal to their sum. Measure with the protractor the number of degrees in the original angles and in the sum, and compare.

14. Construct an angle of 30° . With the bevel construct two other angles equal to it, one on each side of the first, the three bounding lines radiating from the same point. What positions do the outside lines of your figure occupy with respect to each other, and why? Test with an instrument.

15. Construct an angle of 60° . With the bevel construct five other angles equal to it, each adjacent to the preceding, the bounding lines all radiating from the same point. What positions do the first and last lines of these angles occupy with respect to each other, and why?

16. In the figure of the preceding question, if O be the point from which the lines radiate, measure off with the dividers on these lines equal lengths, OA , OB , OC , OD , OE , OF . What do you observe as to the lengths AB , BC , CD , DE , EF , FA ?

17. Fold a piece of paper so as to get a straight crease. Fold the crease over on itself. How many degrees in each of the four angles so obtained, and why?

18. With a needle mark two points. Join them, using ruler and a fine pencil. Turn the ruler over to the other side of the two points and again join them. What quality in the ruler may you test in this way?

19. At points on your paper some distance from one another, construct two angles, as nearly as you can judge, equal. Test with an instrument the correctness of your judgment.

20. Through what angle does the minute-hand of a clock move in 20 minutes? Through what angle does the hour-hand move in the same time?

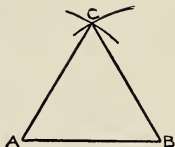
21. Describe a circle, and, supposing it intended for the face of a clock, mark the points where the usual Roman numerals should be placed.

22. One side of a piece of paper being a straight line, tear the remaining boundary into any irregular shape. With your protractor convert this paper into a protractor, so as to mark angles at intervals of 10° , the markings being on the irregular edge of the paper.

CHAPTER II.

Construction of Triangles.

1. Take a line **AB** of any length. First with **A** as centre, then with **B** as centre, and in both cases with the same radius **AB**, describe portions of circles so that they intersect, as indicated, at **C**. Then the three lines **AB**, **BC**, **CA** are all equal. The triangle **CAB**, which has thus all its sides equal, is called an **equilateral** triangle.



Adjust the bevel to each of the angles of this triangle, and compare their magnitudes.

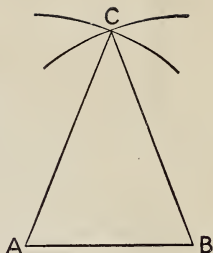
Construct equilateral triangles whose sides are 14, 21, 30, 40 . . . sixteenths of an inch.

Apply the bevel to all the angles of these triangles, and compare their magnitudes.

Cut accurately any one of these equilateral triangles from the paper, and, clipping off the angles, fit them on one another, and on the angles of the other equilateral triangles, so as to compare their magnitudes.

The result of our observations is that **the angles in an equilateral triangle are equal to one another, and are equal to the angles in any other equilateral triangle.**

2. Take a line **AB** of, say, 1 inch in length, and with centres **A** and **B** describe portions of circles intersecting as indicated at **C**, each circle having the same radius, say $1\frac{1}{2}$ inches. Draw lines from **C** to **A** and **B**. Then the triangle **CAB** has two sides equal. A triangle with two of its sides equal is called an **isosceles** triangle.



Adjusting the bevel to the angles **CAB** and **CBA**, compare their magnitudes.

Compare also the sizes of these angles by accurately cutting the triangle out of the paper, and placing the triangle reversed in the vacant space left in the paper, so that the angle **B** rests in the space **A**.

Compare also the sizes of these angles by folding the triangle along the line from **C** to the middle of **AB**.

Construct the following isosceles triangles :

Base 1 in., each side 2 in.

Base 3 in., each side 2 in.

Base $2\frac{1}{2}$ in., each side $2\frac{1}{8}$ in.

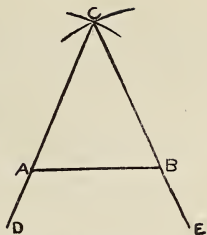
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In each case compare the magnitudes of the angles at the base.

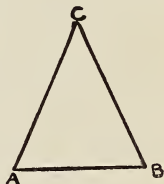
The result of our observations is that **the angles at the base of an isosceles triangle are equal**.

Of course it would follow from this that all the angles in an equilateral triangle are equal, as we have already seen.

Prolong the sides **CA**, **CB**, and adjusting the bevel to the angles **BAD**, **ABE**, on the other side of the base, they will be found to be equal. This may also be reasoned out as follows : The angles on one side of a straight line at any point in it make up 180° . But the angles **CAB** and **CBA** are equal. Therefore the remaining angles **BAD** and **ABE** are also equal.



3. Taking any line **AB**, with the bevel or protractor construct equal angles at **A** and **B**, and produce the bounding lines of these angles to meet in **C**. Then employing the dividers or compasses, compare the lengths of the sides **CA**, **CB**, of the triangle **CAB**.



Construct the following triangles :

Base 1 in., each of angles at base 75° .

Base 3 in., each of angles at base 45° .

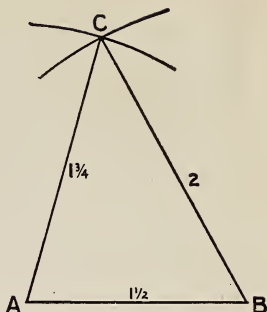
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In each case compare the magnitudes of the sides adjacent to the equal angles.

The result of our observations is that **if two angles of a triangle are equal, the sides opposite to these angles are also equal.**

Of course it will follow that if the three angles of a triangle are equal, the three sides will also be equal, *i.e.*, the triangle will be equilateral.

4. Take a line **AB** of length $1\frac{1}{2}$ inches, and with centres **A** and **B**, and radii $1\frac{3}{4}$ and 2 inches respectively, describe portions of circles, so that they intersect at **C**. Join **CA** and **CB**. We have thus a triangle **CAB** whose sides are unequal, called a **scalene** triangle.



With the bevel or protractor, compare the magnitudes of the three angles of this triangle.

What positions do the greatest, intermediate, and least angles occupy with regard to the greatest, intermediate, and least sides?

Construct the following triangles :

With sides 3, 5 and 6 inches.

With sides $3\frac{1}{2}$, $4\frac{1}{2}$ and $2\frac{1}{2}$ inches.

.

In each case measure the angles of the triangle drawn, and note their positions with regard to the greatest, intermediate, and least sides.

The result of our observations is that, **the angles of a scalene triangle are unequal, and that the greatest, intermediate, and least angles are opposite respectively the greatest, intermediate and least sides.**

This fact may also be stated as follows :—

If one side of a triangle be greater than another, the angle opposite the greater side is greater than the angle opposite the less.

At **A** in the line **AB** construct the angle **BAD** of 40° , and at **B** the angle **ABC** of 120° . Produce **AD**, **BC** to meet in **E**. Compare the length of the side **EA** with that of **EB**. Note which side has the greater angle opposite it.

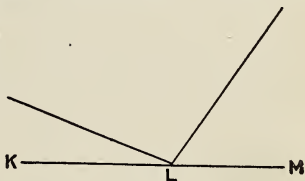
Repeat the same experiment making :—

- (1) The angle **BAD** 80° and the angle **ABC** 20° .
- (2) The angle **BAD** 90° and the angle **ABC** 50° .

The result of such observations will be found to be that **if one angle of a triangle be greater than another, the side opposite the greater angle is greater than the side opposite the less.**

Verify this result by drawing several other triangles and examining as above.

5. Draw any equilateral triangle, any isosceles triangle, any scalene triangle. In the case of each of these triangles, lay off, with the bevel, three angles adjacent to one another, equal to the three angles of each triangle in the way indicated in the figure. Determine the position of the first and last lines, **LM** and **LK**, with respect to each other.

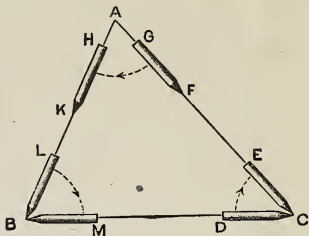


The result of such an examination will be found to be that the lines **KL** and **LM** are in the same straight line, *i.e.*, **the sum of the three angles in each of these triangles is two right angles or 180° .**

Verify this statement by drawing several triangles. Measure the three angles in each with the protractor, and find their sum.

A good general proof of this statement is given below :—

Construct a triangle ABC , and place a pencil in the position DC . Turn the pencil through the angle BCA , in the direction indicated by the arrow head, to the position EC . Slide it along CA , towards A , to the position FG , and turn it through the angle CAB , to the position HK . Slide it along AB to the position BL , and turn it through the angle B , to the position BM .



The pencil has rotated through all the angles of the triangle. But in its final position BM it points in a direction just opposite to its first position DC , and therefore must have rotated through 180° . Hence **all the angles of this (which is any) triangle must together equal 180° , or two right angles.**

What will be the magnitude of each angle of an equilateral triangle?

6. Draw any triangle ABC . Measure carefully the sides AB , AC , and BC . Add the lengths of AB and AC together and compare the result with BC . Add the lengths of AB and BC together and compare the result with AC . Add the lengths of AC and BC together and compare the result with AB .

Repeat the same experiment with several triangles and note your result. The result of such observations will be found to be that **any two sides of a triangle are together greater than the third side.**

Verify this statement by attempting to draw triangles with sides of 2, 3 and 6 inches, 2, 4, and 6 inches.

Summary.

1. An equilateral triangle is one which has its three sides equal. Its three angles are also equal, and each angle is equal to 60° .

2. An isosceles triangle is one which has two sides equal. The angles at the base of an isosceles triangle are equal, and if the equal sides be produced, the angles on the other side of the base will also be equal.

3. If two angles of a triangle be equal, the sides opposite these angles will also be equal.

4. A scalene triangle is one which has three unequal sides. The three angles of a scalene triangle are unequal, and the greatest, intermediate, and least angles are opposite respectively the greatest, intermediate, and least sides.

Or, if one side of a triangle be greater than another, the angle opposite the greater side is greater than the angle opposite the less.

Also, if one angle of a triangle be greater than another, the side opposite the greater angle is greater than the side opposite the less.

5. The sum of the three angles of any triangle is two right angles or 180° .

6. Any two sides of a triangle are together greater than the third side.

Exercises.

1. At a given point in a straight line, construct an angle of 60° , using only compasses and ruler.

2. Construct an isosceles triangle, and produce the base both ways. What do you note as to the magnitudes of the angles so formed outside the triangle? Give reasons why this should be so.

3. The angle at the vertex of an isosceles triangle is 75° , and each of the equal sides is 2 inches. Construct the triangle. With a protractor, measure the remaining angles of this triangle. Verify your results by calculation.

4. The angle at the vertex of an isosceles triangle is 30° , and the base is two inches long. Construct the triangle, first calculating the magnitude of the angles at the base.

5. Each of the angles at the base of an isosceles triangle is 72° , and the two sides opposite these angles are each 3 inches long. Construct the triangle.

6. Describe an equilateral triangle ABC. On BC describe an equilateral triangle DBC; on CA an equilateral triangle ECA; and on AB an equilateral triangle FAB. What do you observe as to the positions of the lines DC, CE with respect to one another? What other lines show the same peculiarity? Give reasons why this should be so.

7. In the preceding question, mark all the lines that are equal to one another. How many equilateral triangles are there in the figure? Name them.

8. With centre A, outside a straight line, describe a circle of such radius as to cut the line in two points, B and C. What sort of triangle is ABC?

9. In the figure of the preceding question find on the side of BC remote from A, a point D, such that a circle with D as centre can be described to pass through both B and C.

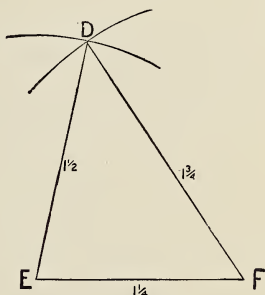
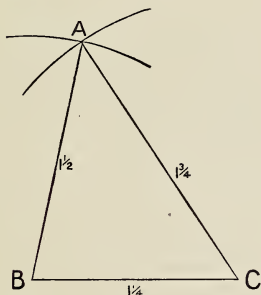
10. B and C being two points in a line, find on either side of the line points K, L; M, N, . . . such that a circle may be described, with any one of them as centre, to pass through B and C. What do you observe as to the positions of K, L, M, N, . . . with respect to one another?

11. Construct a scalene triangle ABC, and on the side of BC away from A, describe a triangle DBC, with $DB=AB$, and $DC=AC$. Join AD. What triangles in the figure are isosceles? What inference can you draw as to the angles BAC and BDC?

CHAPTER III.

Equality of Triangles.

1. Construct two triangles, each with sides of lengths $1\frac{1}{4}$, $1\frac{1}{2}$ and $1\frac{3}{4}$ inches, as indicated in the adjacent figures.



Adjust the bevel, or the protractor, to the angle **A**, and also to the angle **D**, and carefully compare the magnitudes of these angles. In like manner compare the magnitudes of the angles **B** and **E**, and also the magnitudes of the angles **C** and **F**.

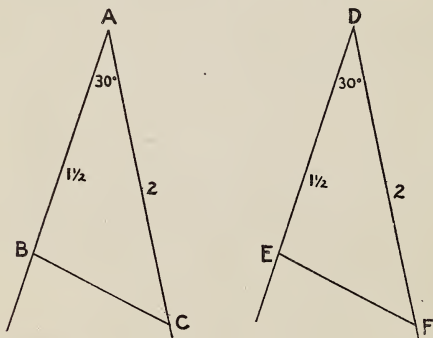
Next cut both triangles from the paper, and place one triangle upon the other so that the corresponding angular points coincide. From this superposition what conclusion do you draw as to the areas of the triangles?

Repeat the same construction, measurement, and superposition with two triangles whose sides are 4, 2 and $4\frac{1}{2}$ inches.

The result of our observations in these cases is that **if two triangles have their sides equal, the angles which are opposite to equal sides are equal and the areas are equal.** In other words two such triangles are the same triangle in different positions.

Another way of stating the fact is to say that if the sides of a triangle are fixed, the angles are fixed, and the area is fixed.

2. Construct two angles, **BAC** and **EDF**, each of 30° . On sides of these angles measure off distances **AB** and **DE**, each of length $1\frac{1}{2}$ inches; and also distances **AC** and **DF**, each of length 2 inches. Join **BC** and **EF**, thus forming two triangles, **ABC** and **DEF**.



Adjust the bevel, or protractor, to the angle **B**, and also to the angle **E**, and carefully compare the magnitudes of these two angles. In like manner compare the magnitudes of the angles **C** and **F**. With the dividers compare the magnitudes of the sides **BC** and **EF**. Further, cut one triangle from the paper, and place it

upon the other. From this superposition what conclusion do you draw as to the areas of the two triangles?

Repeat the same construction, measurement, and superposition with the following triangles :

Two whose sides are $1\frac{3}{4}$ and $2\frac{1}{4}$ inches, and included angle 30° .

Two whose sides are $1\frac{1}{2}$ and 2 inches, and included angle 135° .

.

The result of our observations in all these cases is that **if two triangles have two sides in each equal, and the angles included by these two sides equal, then the remaining sides are equal, and the angles opposite to equal sides are equal, and the triangles are equal in area.** In other words two such triangles are the same triangle in different positions.

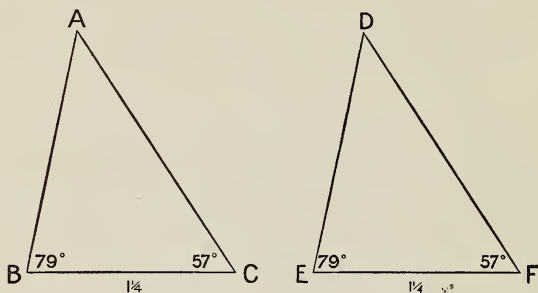
Another way of stating the fact is to say that if two sides and the included angle of a triangle are fixed, the remaining side and angles are fixed, and the area is fixed.

3. Since, as we have shown in Chapter II, the three angles of any triangle together make 180° , therefore the sum of the three angles of any one triangle equals the sum of the three angles of any other triangle.

It follows that **if two triangles have two angles in the one equal to two angles in the other, the third angle in one triangle is equal to the third angle in the other.**

4. Construct two triangles, ABC , DEF , each with base $1\frac{1}{4}$ inch, and angles at the base 79° and 57° .

It follows, from 4, that the remaining angles at **A** and **D** are equal, each being 44° . Putting the points of the dividers on **A** and **B**, and carrying the dividers, so adjusted, to **DE**, compare the magnitudes of **AB** and **DE**. In like manner compare the magnitudes of **AC** and **DF**.



Next, cutting one of the triangles from the paper, place it upon the other. From this superposition what conclusion do you draw as to the areas of the triangles?

Repeat the same construction, measurement, and superposition with the following triangles:

Two whose bases are $1\frac{3}{4}$ in., and angles adjacent to base 38° and 110° .

Two whose bases are $3\frac{1}{2}$ in., and angles adjacent to the base 49° and 95° .

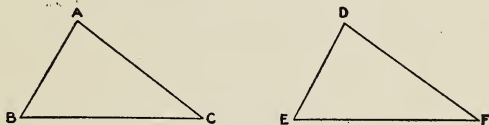
.

The result of our observations in all these cases is that **if two triangles have their bases equal, and angles adjacent to the bases equal, the remaining angles are equal, and the sides**

opposite to equal angles are equal, and the areas are equal. In other words they are the same triangle in different positions.

Another way of stating the fact is to say that if a side of a triangle and the angles adjacent to this side are fixed, then the remaining angle and sides are fixed, and area is fixed.

5. It must be observed in connection with the results obtained in section 4, that if two triangles have **one side and any two angles of one equal respectively to one side and the corresponding angles of the other, then the triangles will be equal in all respects**; for if two angles of one triangle are equal to two angles of the other, the remaining angles will also be equal. That is, we shall have one side and its adjacent angles in one triangle equal to one side and its adjacent angles in the other triangle.



E.g., in the triangles **ABC** and **DEF**, suppose the side **BC** = side **EF**, the angle **ABC** = the angle **DEF**, and the angle **BAC** = the angle **EDF**.

It follows that the angle **ACB** = the angle **DFE** (sec. 3).
Now, we know that **BC** = **EF**,

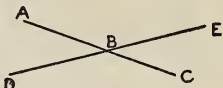
that the angle **ABC** = the angle **DEF**,
and that the angle **ACB** = the angle **DFE**.

Therefore the triangles are equal in all respects.

6. Draw **AC** and **DE** any two straight lines cutting one another at **B**.

The angles ABD and CBE are called **vertically opposite angles**, as also are the angles ABE and DBC .

Using the bevel, what do you observe with reference to the magnitudes of the **vertically opposite angles** ABD and CBE ?



What with reference to the magnitudes of the angles ABE , DBC ?

Draw other intersecting straight lines and note the magnitudes of vertically opposite angles.

We may demonstrate the relation between such angles as follows :

The angle $ABD = 180^\circ - \text{the angle } ABE$.

The angle EBC also $= 180^\circ - \text{the angle } ABE$.

$\therefore$ Angle $ABD = \text{angle } EBC$.

Show similarly that angle $ABE = \text{angle } DBC$.

Hence, if two straight lines cut one another, the vertically opposite angles are equal.

Summary.

1. By saying that two triangles are equal in all respects we mean that, the three sides of one triangle are respectively equal to the three sides of the other, the three angles of one triangle are respectively equal to the three angles of the other, and the triangles are equal in area.

2. Two triangles will be equal in all respects if:—

(1) Three sides of one are respectively equal to three sides of the other.

Or (2) Two sides and the contained angle of one are respectively equal to two sides and the contained angle of the other.

Or (3) One side and two angles of one are equal respectively to one side and the corresponding angles of the other.

3. By saying that a triangle is fixed or determined we mean that from the information given, that triangle and no other may be drawn.

4. A triangle is fixed or determined if:—

(1) Three sides are given.

Or (2) Two sides and contained angle are given.

Or (3) Two angles and one side are given (the side being given either adjacent to the given angles or opposite one of them).

5. If two triangles have two angles of one respectively equal to two angles of the other, the third angle of the one is equal to the third angle of the other.

6. If two straight lines cut each other, the vertically opposite angles are equal.

.Exercises.

1. The sides of a triangle are $1\frac{1}{2}$, 2 and $2\frac{1}{2}$ inches. Construct the triangle; and with the protractor measure the angles to the nearest degree.

2. Two sides of a triangle are $2\frac{1}{2}$ and $1\frac{1}{2}$ inches, and the included angle is 47° . Construct the triangle; and measure the remaining side to the nearest sixteenth of an inch, and the remaining angles to the nearest degree.

3. Two sides of a triangle are $5\frac{1}{2}$ and $6\frac{1}{2}$ inches, and the included angle is 54° . Construct the triangle; and measure the remaining side to the nearest sixteenth of an inch, and the remaining angles to the nearest degree.

4. Two angles of a triangle are 107° and 27° , and the side adjacent to them is 2 inches. Construct the triangle; and measure the remaining angle to the nearest degree, and the remaining sides to the nearest sixteenth of an inch.

5. Two angles of a triangle are 53° and 66° , and the side adjacent to them is 4 inches. Construct the triangle; and measure the remaining angle to the nearest degree, and the remaining sides to the nearest sixteenth of an inch.

6. The sides of a triangle are 4, 6 and 7 inches. Construct the triangle; and measure the angles to the nearest degree.

7. Two sides of a triangle are $3\frac{1}{2}$ and $2\frac{3}{4}$ inches, and the included angle is 58° . Construct the triangle; and measure the remaining side to the nearest sixteenth of an inch, and the remaining angles to the nearest degree.

8. Two angles of a triangle are 30° and 128° , and the side adjacent to them is $2\frac{1}{2}$ inches. Construct the triangle; and measure the remaining angle to the nearest degree, and the remaining sides to the nearest sixteenth of an inch.

9. Two angles of a triangle are 35° and 120° , and the side opposite the larger of these angles is $2\frac{1}{2}$ inches long. Construct the triangle.

10. One side AB of a triangle ABC is $1\frac{1}{4}$ inches long, the angle ABC is 40° , and the angle ACB is 70° . Construct the triangle, and explain why the side AB is equal to the side BC.

11. Two lines AB and CD intersect in E, and, with the dividers, AE and EB are taken equal to one another, and also CE and ED equal to one another. Show, without taking any other measurements, that the triangles EAC and EBD are equal in all respects. What lines and angles should therefore be equal? Verify by measurement.

12. A triangle ABC is described, and on the other side of BC a triangle DBC is constructed, with $DB = AB$ and $DC = AC$. Show that the two triangles are equal in all respects. What angles of this figure should therefore be equal? Verify by measurement.

13. A triangle ABC is described, and on the other side of BC the triangle DBC is constructed, with $DB = AC$ and $DC = AB$. Show that the two triangles are equal in all respects. What angles should therefore be equal? Verify by measurement.

14. A triangle ABC is described, and on the same side of BC another triangle DBC is described, with $DB = AC$ and $DC = AB$. Show that triangles ABC and DCB are equal in all respects. What angles should be equal as a result? Verify by measurement.

15. If in the preceding exercise AD is joined, and BA and CD produced to meet in E, what are the triangles EAD, EBC? Give reasons.

16. From two lines diverging from A, equal lengths AB, AC are cut off, also equal lengths AD, AE. CD and BE are joined. Show that the triangles ABE and ACD are equal in all respects. What lines and angles should therefore be equal? Verify by actual measurement.

17. Two circles have the same centre O . AOB is the diameter of one, and COD the diameter of the other. AC and BD are joined. What triangles in the figure are equal in all respects? Why? What lines and angles should therefore be equal? Verify by actual measurement.

18. Describe ABC , DBC , two isosceles triangles on the same base BC , but on opposite sides of it. How does AD divide the angles BAC , BDC ? Why?

19. If in the preceding question AD cuts BC in E , show that $BE = EC$, and that AD is perpendicular to BC . (Review definition of perpendicular.)

20. Construct an equilateral triangle ABC . At B and C construct equal angles GBC , GCB . Join AG . How does AG divide the angle BAC ? Why?

21. Draw a triangle with angles 30° , 70° , 80° . How many such triangles may be drawn? Is a triangle fixed if its three angles are given? If two triangles have three angles of one respectively equal to three angles of the other, are the triangles necessarily equal in all respects?

CHAPTER IV.

Bisection of Lines and Angles. Perpendiculars.

1. To bisect a straight line.

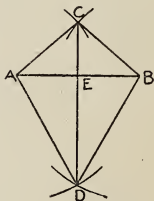
Suppose **AB** the line to be bisected. With **A** and **B** as centres describe portions of circles with equal radii intersecting at **C**, and with the same centres describe portions of circles with equal radii, intersecting at **D**. Then if **CD** be drawn, it bisects **AB** at right angles.

For, using the dividers, it will be found that **AE** and **EB** are equal; and, using the protractor or set-square, all the angles at **E** will be found to be 90° .

Or again, we would conclude that **AE** and **EB** are equal, and that the angles at **E** are right angles, from the **symmetry** of the figure with respect to the line **CD**—the figure on one side of this line being just the same as the figure on the other side, but turned in the opposite direction.

Or again we may reason out the equality of **AE** and **EB** and that the angles at **E** are right angles as follows:

In the triangles **ACD**, **BCD**; **AC** = **BC**, **AD** = **BD** and **CD** is common to both triangles. Therefore the triangles are equal in all respects, so that the angle **ACD** = the angle **BCD**.



Again in the triangles **ACE** and **BCE**, $AC = BC$ and **CE** is common to both triangles, also the angle **ACE** = the angle **BCE** (proved). Therefore the triangles are equal in all respects, so that $AE = EB$, and the angle **CEA** = the angle **CEB**; therefore each of these angles is a right angle.

In practice it is not necessary to draw the lines **AC**, **BC**, **AD**, **BD**, **CD**. Having found the points **C** and **D**, placing the ruler on these points, we may mark the point **E** in **AB**.

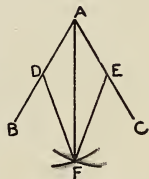
Subsequently, when the subject of parallel lines comes to be dealt with, another and possibly readier way of finding the middle point of a line will be given.

A number of exercises should now be given in bisecting lines of different lengths, the dividers being used in each case to determine whether the point reached is the middle point.

It is suggested that the pupil be given exercises in estimating with the eye the middle points of lines of various lengths, these points being afterwards accurately determined by geometrical construction.

2. To bisect an angle.

Let **BAC** be the angle. Place one of the points of the dividers or compasses at **A**, and mark off equal lengths **AD**, **AE** in **AB** and **AC**. With centres **D** and **E** describe portions of circles with equal radii, intersecting at **F**. Then drawing **AF**, the angle is bisected by it.



For, adjusting the bevel to either of the angles at **A**, it will be found equal to the other.

Or again, we would conclude that the angles at **A** are equal from the symmetry of the figure with respect to

the line **AF**—the figure on one side of this line being just the same as the figure on the other side, but turned in the opposite direction.

Or again, we may prove the equality of the angles as follows : The triangles **DAF**, **EAF** have their sides equal. Hence (Ch. III., 1) the angles **DAF**, **EAF** are equal.

In practice it is not necessary to draw the lines **DF**, **EF**.

A number of exercises should be given in bisecting angles of various magnitudes, the bevel being used in each case to determine whether the bisection is accurate.

The protractor may also be used for bisecting angles.

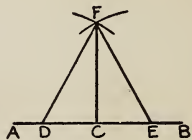
It is suggested that the pupil be given exercises in estimating with the eye the bisecting lines of a number of angles, the bisection being afterwards accurately reached by geometrical construction.

Greater accuracy is likely to be secured in bisecting an angle, by making **AD**, **AE** and **DF**, **EF** of considerable length. The point **F** is then remote from **A**, and any trifling error in locating the exact point where the circles intersect, has less effect on the angle at **A** through being on the circumference of a large circle (radius **AF**).

3. From a point in a line to draw a line at right angles to it.

If **C** be the point in **AB** from which the perpendicular is to be drawn, place one point of the dividers or compasses at **C**, and mark off equal lengths **CD** and **CE**.

Then with centres **D** and **E** describe portions of circles with equal radii, intersecting at **F**. Draw **FC**: it is perpendicular to **AB**.



For, applying the set-square or protractor, the angles at **C** will be found to be right angles.

Or again, from the symmetry of the figure with respect to **CF**, we may conclude that the angles at **C** are right angles.

Or again, since the sides of the triangles **DCF**, **ECF** are equal, therefore (Ch. III., 1) these triangles are equal in all respects, and the angles at **C** are equal. Hence the angles at **C** are right angles.

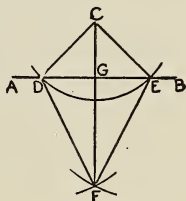
In practice the lines **FD** and **FE** need not be drawn.

A number of exercises should be given in drawing lines at right angles to others from points in the latter, the correctness of the constructions being tested by using the set-square or protractor.

In future, in the various constructions that are to be made, where a line is to be drawn at right angles to another from a point in the latter, the set-square or protractor should in general be used instead of this construction.

4. To draw a line perpendicular to another from a point without the latter.

Let **C** be the point without **AB** from which the perpendicular is to be drawn to **AB**. With **C** as centre describe a circle cutting **AB** in **D** and **E**. With **D** and **E** as centres describe portions of circles with equal radii, intersecting at **F**. Join **CF**, cutting **AB** in **G**. **GG** is the perpendicular from **C** on **AB**.



For, applying the set-square or protractor, the angles at **G** will be found to be right angles.

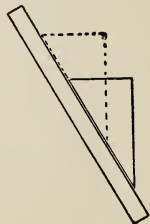
Or again, from the symmetry of the figure with respect to **CF**, we may conclude that the angles at **G** are right angles.

Or again, since the sides of the triangles **DCF**, **ECF** are equal, therefore (Ch. III., 1) the angles at **C** are equal. Also, since in the triangles **DCG**, **ECG** the angles at **C** are equal, and the sides about these angles equal, therefore (Ch. III., 2) these triangles are equal in all respects, and the angles at **G** are equal. Hence the angles at **G** are right angles.

In practice the lines **CD**, **CE**, **FD**, **FE**, **GF** need not be drawn.

A number of exercises should be given in drawing lines perpendicular to others from points without the latter, the correctness of the constructions being tested by using the set-square or protractor.

5. In future, where a line is to be drawn perpendicular to another from a point without the latter, the set-square or protractor should in general be used instead of the preceding construction. When for this purpose the protractor is used, the edge of the ruler is to be placed over the centre-point of the protractor and over the 90° mark; the base of the protractor is then to be slid along the line until the edge of the ruler is over the given point without the line. The centre-point of the protractor then marks the foot of the perpendicular on the line.

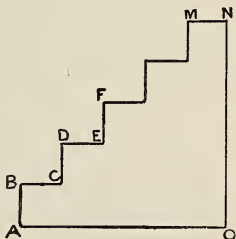


The annexed diagram illustrates how, by sliding the set-square along the ruler, lines may be drawn parallel to each other; and also how a line may be drawn perpendicular to

another from any point, whether the point be without or on the latter line. In drawing a perpendicular to a line by placing an edge of the right angle of set-square against the latter, we are often unable to bring the perpendicular up to the line by reason of the right angle of the set-square having become rounded through use.

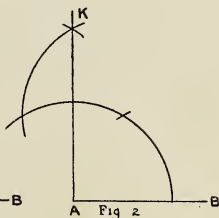
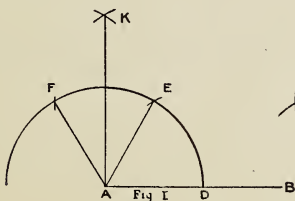
Sometimes a convenient way of drawing a perpendicular through a point is to draw a perpendicular in any position, and then a parallel to this through the given point, the former perpendicular being afterwards erased.

Draw a number of equal and perpendicular lines $AB, BC, CD, \dots MN$, and finally draw AO perpendicular to AB , and NO perpendicular to MN , as indicated in the figure. The accuracy of the series of constructions may be tested by the conditions that AO is both equal and perpendicular to NO .



6. Another convenient method of drawing a perpendicular to a given line from a point within it is given below :—

Let AB be the given straight line, and A any point within it.



With centre **A** and any radius, describe an arc cutting **AB** at **D**.

With centre **D** and same radius, describe an arc cutting first arc at **E**.

With centre **E** and same radius, describe an arc cutting first arc again at **F**.

Join **EA**, **FA** and bisect angle **EAF** by **AK**. Then **AK** is at right angles to **AB**.

For the angle **EAD** = 60° , and also angle **EAF** = 60° .

Therefore angle **KAE** = 30° .

Therefore angle **KAD** = 90° .

In actual practice the perpendicular may be constructed, using only lines shown in Fig. 2.

Summary.

1. To bisect a straight line at right angles.
2. To bisect an angle.
3. From a point in a line to draw a line at right angles to it (two methods).
4. To draw a line perpendicular to another from a point without the latter.
5. Use of set square and ruler to draw lines parallel and perpendicular to each other.

Exercises.

1. What is meant by the distance of a point from a line?
2. **AOB** is any angle and **OC** bisects it. **D** is any point in **OC**, and perpendiculars **DE**, **DF** are drawn from **D** on **OA** and **OB** respectively. Show that the triangles **OED**, **OFD** are equal in all respects. What does this show with regard to the lines **DE** and **DF**; the angles **ODE**, **ODF**?
3. In the preceding exercise **D** is said to be equidistant from **AO** and **OB**. Is this true with regard to any other point on **OC**? How would you find a line so that any point on it is equidistant from two given lines **OA** and **OB**?

4. Construct a triangle ABC , and find a point in the base BC such that the perpendiculars from it on AB and AC are equal.

5. AB and CD are two straight lines which are not parallel. How would you find a line, any point on which is equidistant from both of them?

6. AB is any straight line. Bisect it at right angles by the line CD . What do you observe as to the distances of any point on CD from A and B ?

7. Taking any two points, A and B , in the plane of the paper, draw a line such that the distances from any point on it to A and B are equal.

8. Find a point equidistant from two given points A and B , and one inch from a third given point C . Is it always possible to do this?

9. In a given straight line find a point which is equidistant from two given points not lying in the line.

10. Find a point which is equidistant from three given points *not* lying in one straight line.

11. Find a point which is equidistant from three given straight lines which are not parallel.

12. Draw an acute angle BAC . From a point O outside the angle draw perpendiculars OD , OE to AB and AC respectively. Show that the angle BAC is equal to the angle DOE .

13. Construct a triangle one of whose angles is a right angle. How many degrees do you find in the sum of the other two angles? Give reason.

14. Construct a triangle ABC with C a right angle. At C make the angle BCD equal to the angle CBA , CD meeting AB in D . What do you note as to the magnitudes of the lines DA , DB , DC ? Give reason.

15. ABC is an isosceles triangle with $AB = AC$. Produce BA to D , making AD equal to AB or AC . Join CD . What is the magnitude of the angle BCD ? Give reason.

16. Does exercise 15 suggest a method of drawing from a point at the end of a given line, another line at right angles to it?

17. Construct an equilateral triangle ABC , with side two inches. Draw AD to the bisection of the base BC . How many degrees are there in each of the angles of the triangle ABD ?

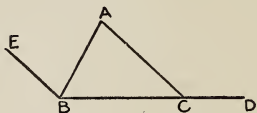
CHAPTER V.

Respecting Angles of a Triangle.

1. We have seen (Ch. II., 5) that the sum of the three angles of any triangle is two right angles, or 180° .

Definition: In any rectilineal figure, an *exterior angle* is an angle contained by any side and an adjacent side produced.

Produce the side **BC** to **D**.
With the bevel or protractor lay off the angle **ABE** equal to the angle at **A**. Using the bevel, examine now the relation existing between the angle **EBC**, which is equal to the sum of the angles **A** and **B** of the triangle, and the *exterior angle* **ACD**.



Repeat the construction and examination in the case of a triangle of different shape, say one in which the angle at **B** is an obtuse angle.

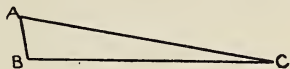
If the constructions and measurements have been accurately made, it will be found that **the exterior angle (ACD) is equal to the sum of the two interior and opposite angles at A and B.**

We may show that this is always the case as follows: The three angles of the triangle make up 180° ; but the angles **ACB**, **ACD** also make up 180° ; hence the angle **ACD** must be equal to the sum of the angles **A** and **B**.

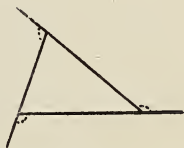
2. Of course, since the exterior angle of any triangle is equal to the sum of the two interior and opposite angles, it follows that **the exterior angle of any triangle is greater than either of the interior and opposite angles.**

Also, since the sum of the three angles of any triangle is equal to two right angles, **the sum of any two angles of a triangle is less than two right angles.**

We may, however, by elongating the triangle, make the sum of two of its angles but little short of two right angles. Thus the sum of **A** and **B** will be found, on using the protractor, to be but little less than 180° ; and, by still further removing **C**, we may still further increase their sum.

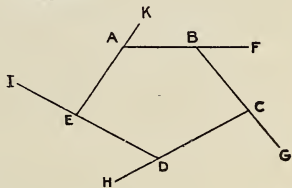


3. Lay off about a point, and adjacent to one another, angles equal to the three exterior angles of the triangle; or, with the protractor, measure the number of degrees in each of these angles. What is their sum? Give reasons for this sum being what it is.



4. Construct any polygon **ABCDE**.

Taking the sides in order produce each as shown in figure. Place your pencil along **AB**, noting in which direction it points. Revolve it through the angle **KAB**, slide it along **AE** to **E** and revolve it through the angle **IEA**, slide along **ED** and revolve through the angle **EDH**, slide along **DC**



and revolve through the angle **DCG**, slide along **CB** and revolve through the angle **CBF**. Note the direction in which the pencil now points. Through how many degrees has it revolved? What must be the sum of all the exterior angles of the figure? Verify by measuring these angles and finding their sum.

Repeat the same experiment with figures of four sides, of seven sides, etc., and note your results.

The result of such observations will be found to be that **if any polygon have its sides produced in order, the sum of the exterior angles so formed will be four right angles or 360° .**

5. In the figure **ABCDE** above, what is the sum of each exterior angle and its adjacent interior angle? What is the sum of all the exterior angles and all the interior angles of this figure? Knowing that the sum of the exterior angles is four right angles or 360° , what must be the sum of all the interior angles? Find in the same way the sum of all the interior angles of a seven-sided figure, a nine-sided figure.

6. A figure is said to be **regular** if all its sides and also all its angles are equal.

Ex. To find the magnitude of each exterior angle and also each interior angle of a regular hexagon (six sides).

Since the six exterior angles = 360° .

Therefore each exterior angle = 60° .

Therefore each interior angle = $180^\circ - 60^\circ = 120^\circ$.

Find in the same way the magnitude of each exterior and interior angle of:

- (1) A regular pentagon (5 sides).
- (2) A regular heptagon (7 sides).
- (3) A regular octagon (8 sides).
- (4) A regular decagon (10 sides), etc.

7. The following facts obtained in Chapter II. will be of use in connection with the exercises on this chapter.

(1) If one side of a triangle be greater than another, the angle opposite the greater side will be greater than the angle opposite the less.

(2) If one angle of a triangle be greater than another, the side opposite the greater angle will be greater than the side opposite the less.

Summary.

1. If one side of a triangle be produced, the exterior angle so formed is equal to the sum of the two interior and opposite angles.

2. The exterior angle of a triangle is greater than either of the interior and opposite angles taken separately.

3. Any two interior angles of a triangle are together less than two right angles.

4. If the sides of any rectilineal figure be produced in order, the sum of the exterior angles so formed will be four right angles, or 360° . From this fact may be deduced the sum of all the interior angles of any rectilineal figure.

5. How to find the magnitude of the exterior and interior angles of a regular figure of a given number of sides.

Exercises.

1. Construct a quadrilateral figure. Join one pair of opposite corners, and hence show that the sum of the interior angles of the quadrilateral is four right angles. Verify by measuring the angles and finding their sum. Can you show this in any other way?

2. What is the magnitude of each interior angle of :—

(1) A regular figure of five sides ?

(2) A regular figure of ten sides ?

(3) A regular figure of fifteen sides ?

3. Construct an isosceles triangle ABC ($AB = AC$). In AB take any point D . With the dividers or compasses determine whether D is nearer to B or to C . Give reason.

4. A triangle can have only one angle either equal to or greater than a right angle, *i.e.*, at least two of the angles of a triangle must always be acute angles. Why?

5. AB is any straight line, and O is a point outside it. OC is drawn perpendicular to AB. Show that OC is shorter than any other line that may be drawn from O to AB. Draw other lines from O to AB. What do you observe as to their magnitudes according as they approach to or recede from the perpendicular?

6. Which is the greatest side of a right-angled triangle, and why?

7. ABC is a triangle, and O is a point within it. Is the angle BOC greater than, equal to, or less than the angle BAC? Give reasons. (Join AO and produce to E.)

8. Can more than two equal straight lines be drawn to a straight line from a point without it?

9. In how many points can a circle cut a straight line?

10. In the triangle ABC can you find a point D, such that AD is equal to or greater than the greater of the sides AB, AC?

11. In any triangle can you find a point such that the distance from it to any one of the angles is equal to or greater than the greatest of the sides?

12. Describe two circles with the same centre, *i.e.*, concentric. Take a point A on the circumference of one, and a point B on the circumference of the other. When will the line AB be least? Give reasons.

CHAPTER VI.

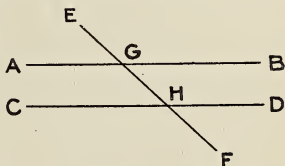
Parallel Lines.

1. Parallel lines were defined to be such as have the same direction. Thus the lines in the figure, though differing in position, have all the same direction, and are parallel.



2. With the parallel rulers draw two parallel lines **AB** and **CD**, and draw another line **EF** cutting **AB** in **G** and **CD** in **H**.

The angle **EGB** is said to be an **exterior angle**, and with respect to it, the angle **GHD** is said to be the **interior and opposite angle**. Point out three



other exterior angles in this figure with their corresponding interior and opposite angles.

The angles **AGH** and **GHD** are called **alternate angles**. Point out another pair of alternate angles in this figure.

BGH and **GHD** are spoken of as the two **interior angles** on one side of the line **EF**. What are the two interior angles on the other side of this line?

3. Draw any two parallel straight lines, and a line cutting them. With the bevel or protractor, compare the magnitude of any exterior angle with that of its

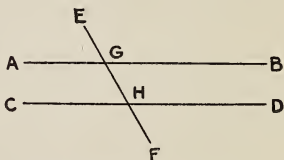
corresponding interior and opposite angle. Try the same experiment with several pairs of parallel straight lines and note your result.

The result of such observations will be that **the exterior angle is equal to its corresponding interior and opposite angle.**

4. Compare in the same way the magnitudes of any pair of alternate angles. They also will be found to be equal.

This fact may easily be deduced from the fact already obtained, as follows :—

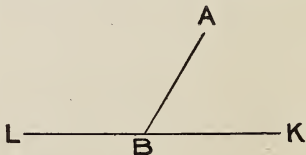
In the fig., if **AB** and **CD** are parallel, the angle **EGB** (ext. angle) = the angle **GHD** (int. and opp. angle). But the angle **EGB** = vertically opposite angle **AGH**;



$\therefore$ the angle **AGH** = the angle **GHD**. Prove in the same way that the angle **BGH** = the angle **GHC**.

It is understood, then, that if a straight line cut two parallel straight lines, **the alternate angles will be equal.**

5. Draw two parallel straight lines and a third line cutting them. Measure with the protractor the two interior angles on one side of the cutting line. Find their sum. Lay them off with the bevel side by side, as shown in the figure.



Note the position of lines **LB** and **BK** with regard to one another. What do you conclude as to the sum of these two angles?

Repeat the same experiments with the interior angles on the other side of the cutting line. Draw several pairs of parallel lines and test in the same way.

The result of such experiments shows that if a straight line cuts two parallel straight lines, **the two interior angles on the same side of the cutting line will together equal two right angles, or 180° .**

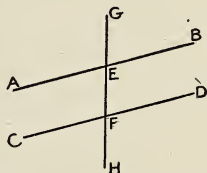
We may also prove this thus:

$$\angle GEB = \angle EFD;$$

$$\therefore \angle GEB + \angle BEF = \angle EFD + \angle BEF.$$

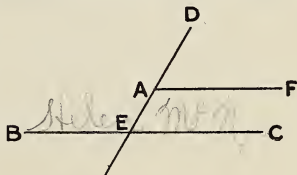
$$\text{But } \angle GEB + \angle BEF = 2 \text{ rt. angles;}$$

$$\therefore \angle EFD + \angle BEF = 2 \text{ rt. angles.}$$



6. Draw any straight line **BC** and another straight line **DE** cutting **BC** at **E**.

At **A** in the straight line **DE** make an angle **DAF** = angle **AEC** (using bevel or protractor). Test **AF** and **BC** with parallel rulers, or set-



squares, to determine whether they are parallel or not.

Repeat the same experiment, making several different figures.

The result of such observations will be that **if one straight line cut two other straight lines so as to make the exterior angle equal to the interior and opposite angle, then these two straight lines will be parallel.**

This may also be reasoned out as follows:

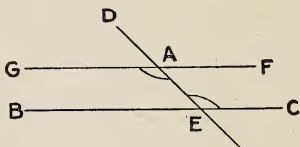
The line **AF** deviates from **DE** by the angle **DAF**; the line **BC** from **DE** by the angle **DEC**, but angle **DAF** = angle **DEC**;

$\therefore$ lines **AF**, **BC** deviate from **DE** by the same amount;

$\therefore$ **AF**, **BC** have the same direction; $\therefore$ they are parallel.

7. Draw any straight line **BC** and another line **DE** cutting **BC** at **E**.

At **A** in the line **DE** make the angle **GAE** equal the angle **AEC** (using bevel or protractor). With set-squares or parallel rulers determine whether or not **GA** and **BC** are parallel. Draw several figures experimenting with each in the same way.



The result of such experiments will show that if one straight line meets two other straight lines so as to make the alternate angles equal, these two straight lines are parallel.

This may be reasoned out as follows:

$$\angle GAE = \angle AEC.$$

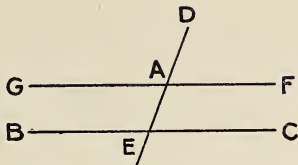
But $\angle GAE = \angle DAF$ (vertically opposite angles);

$$\therefore \angle DAF = \angle AEC.$$

That is, the exterior angle = the interior and opposite angle;

$\therefore$ **GF** and **BC** are parallel. (Sec. 6.)

8. Draw any straight line **BC**, and another straight line **DE** cutting it at **E**. With the protractor measure the angle **AEC**.



Subtract your result from 180° .

At **A** in the line **DE** make the angle **FAE** (on the same side of the line **DE** as $\angle AEC$) equal to the remainder obtained from your subtraction. Test lines **AF** and **BC** with parallel rulers and set-squares.

What is the sum of angles **FAE** and **AEC**?

Repeat the same experiment with several different figures.

The results of such experiments show that if one straight line meet two other straight lines so as to make the two interior angles on the same side of the line together equal to two right angles, or 180° , the two straight lines are parallel.

This may also be reasoned out as follows :—

$$\angle FAE + \angle AEC = 180^\circ.$$

$$\text{But } \angle DAF + \angle FAE = 180^\circ;$$

$$\therefore \angle FAE + \angle AEC = \angle DAF + \angle FAE.$$

Take from each $\angle FAE$;

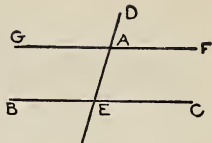
$$\therefore \angle DAF = \angle AEC.$$

That is, ext. angle = int. and opp. angle ;

$\therefore$ **GF** is parallel to **BC** (sec. 6).

9. To draw a straight line through a given point **A** parallel to a given straight line **BC**.

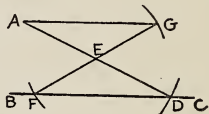
Through **A** draw **DAE**, cutting **BC**, and make the angle **DAF** equal to the angle **AEC**. Then **AF** is parallel to **BC**. **FA** may then be produced, if necessary, to **G**.



Of course we could have drawn **GA** parallel to **BC** by making the angle **GAE** equal to the alternate angle **AEC**.

The line through **A** parallel to **BC** can also be drawn without measuring any angle, as follows :—

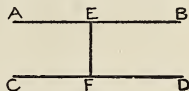
With **A** as centre and radius, say, of 2 inches, describe a portion of a circle cutting **BC** in **D**. Measure off on this a distance **AE** of 1 inch, so that **E** is the middle point of **AD**. With centre **E** and any radius of sufficient length to reach **BC**, describe a portion of a circle cutting **BC** in **F**; and let the diameter of this circle, through **F**, meet the circle again in **G**. Then **AG** is parallel to **BC**. For the sides **AE**, **EG** of the triangle **AEG** are equal to the sides **DE**, **EF** of the triangle **DEF**. Also the angles **AEG**, **DEF** are equal. Hence these triangles are equal in all respects, and the angle **GAE** is equal to the angle **EDF**. **AG** is therefore parallel to **BC**.



A number of exercises should be given pupils in drawing lines through given points parallel to lines in given positions, using both the preceding methods. At the end of each construction the accuracy of the work may be tested with the parallel rulers, or with ruler and set-square (Ch. IV., 5), or by examining whether lines drawn perpendicular to each pair of parallels are equal in length. (See 10 and 11, following.)

For the most part, in future, in drawing parallel lines parallel rulers are to be used, or ruler and set-square (Ch. IV., 5).

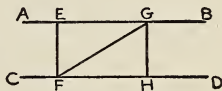
10. A straight line which is perpendicular to one of two parallel lines, is also perpendicular to the other. The truth of this should be tested by drawing with the set-square a line perpendicular to one of the parallels, and examining, with the set-square, whether it is also perpendicular to the other.



Of course this is only a particular case of the truth, that parallel lines have each the same direction with respect to any third line that cuts them.

Or we may *prove* it as follows: If $\angle DFE$ is a right angle, then since $\angle DFE + \angle FEB = \text{two rt. angles}$, $\angle FEB$ must also be a right angle.

11. Two parallel lines are, of course, throughout their lengths at the same distance from one another. For, with the set-square or protractor, draw lines EF , GH , . . . perpendicular to AB and CD . Then, if the dividers be adjusted to the length EF , they will be found to be adjusted to the other lengths GH , . . .



We may *prove* that this is always the case, as follows: EF and GH are parallel to one another because they have the same direction with respect to the third line AB (or CD).

Again, $\angle EGF = \angle GFH$, being alternate angles ;

$\angle EFG = \angle HGF$, " " "

Side FG is common to the two triangles ;

$\therefore$ (Ch. III., 4) $EF = GH$.

We have everywhere illustrations of this. Thus we say that an ordinary board or ruler, whose sides are parallel, is of the same *width* throughout its length.

12. The method of drawing a line parallel to another by sliding the set-square along the ruler (Ch. IV., 5) receives its justification in § 6 of this chapter. The line **DAE** corresponds to the edge of the ruler; the lines **AF**, **EC** to the edge of the set-square in its two positions; and the angles **DAF**, **AEC** to the angle of the set-square in its two positions, the angle of the set-square being of course always the same.

It may be added that, in drawing parallel lines, some prefer the ruler and set-square to parallel rulers. The cost of an instrument is saved. If the edges of ruler and set-square are perfectly straight, the method gives absolutely correct results. Parallel rulers possibly work more rapidly and conveniently.

Summary.

1. If one straight line meet two parallel straight lines, (a) the exterior angle = the int. and opposite angle on the same side of the line.

(b) The alternate angles are equal.

(c) The two interior angles on the same side together equal two right angles or 180° .

2. If one straight line meet two other straight lines so that,—

(a) the ext. angle = int. and opposite angle;

or (b) the alternate angles are equal;

or (c) the two interior angles on the same side of the line together = two right angles or 180° ,

then these two straight lines are parallel.

3. To draw a straight line parallel to a given straight line, through a given point.

4. Parallel straight lines are the same distance from one another throughout their entire length.

Exercises.

1. If AB , CD intersect in O , and $AO=OB$, and $CO=OD$, what position do AD , CB occupy with respect to each other; and what do AC and DB ? Apply tests with instruments. Give reasons, i.e., *proof*.
2. If AB , CD intersect in O , and $AO=OD$, $CO=OB$, what position do AD , CB occupy with respect to each other? Apply tests. Give reasons.
3. Construct a quadrilateral with two sides equal, and the other two parallel and unequal.
4. In the preceding question produce the equal sides to meet, and by applying tests determine the character of the two triangles so formed.
5. The two interior angles on the same side which one line makes with two others are 105° and 70° . Will the two last mentioned lines meet? On which side of the cutting line?
6. A , B are the extremities of the diameter of a circle, and parallel lines AC , BD are drawn, terminated by the circle. What is the relation of AC , BD as to magnitude? Give reasons. (Bisect AB at O and join CO , OD .)
7. A is a point not lying in the straight line BC . From A draw lines AD , AE , AF , . . . to BC , and produce them to K , L , M , . . . making $DK=AD$, $EL=AE$, $FM=AF$, ... What do you observe as to the positions of the points K , L , M , . . . ?
8. In the preceding question find PQR middle points of AD , AE , AF , ... What do you observe as to their positions?
9. Two parallel lines are 3 inches apart, and a point A is taken 2 inches from one line and 1 inch from the other. Lines are drawn through A terminated by the parallels. By measurement determine how these lines are divided at A .
10. Construct a triangle with sides 2, 3 and 4 inches. Bisect the sides and join the points of bisection. What do you observe as to the direction of the sides of the new triangle? What as to magnitudes of its angles and sides?
11. Construct a triangle, and through its angular points, with the parallel rulers draw lines parallel to the opposite sides. Four new triangles are thus constructed. Compare their sides and angles with those of the original triangle, and give results of comparison.

12. Construct a triangle ABC, and through any points D, E, F in the plane of the paper draw lines parallel to BC, CA, AB. Compare the angles of the new triangle with those of the original.

13. If through a point A any two lines be drawn, and through any point B lines be drawn parallel to the former two, show that the angles at A and B are equal. Why?

14. ABC, CDE are two triangles with AB, CD equal and parallel, and also BC, DE equal and parallel. What position do AC, CE occupy with respect to each other?

15. Make an irregular drawing on the paper to represent a pond, or other obstruction, and on opposite sides of it take points A and B. By a line construction about the pond, with measurements, obtain a line at A which if produced would pass through B, without placing the ruler on AB.

16. Draw two lines, both parallel to the same straight line. What is their direction with respect to each other?

17. The side BC of a triangle ABC is produced to D. Bisect the angles BAC, ACD. Can the bisecting lines be parallel to one another?

18. On any line AC as diagonal, construct a quadrilateral ABCD with its opposite sides equal. How are the opposite sides placed with respect to each other? Test and give proof.

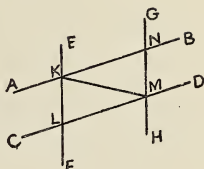
19. Two lines make an angle of 63° with each other. Place a straight line 2 inches long with its ends resting on them, and making an angle of 80° with one of them.

20. Draw two lines perpendicular to the same straight line. What is their direction with respect to each other?

CHAPTER VII.

Parallelograms, Rectangles and Squares.

1. With the parallel rulers, or by other means, draw a pair of parallel lines **AB**, **CD**, and also another pair of parallel lines **EF**, **GH**, inclined to the former pair at any angle.



The figure **KLMN** is called a **parallelogram**, *i.e.*, a parallelogram is a four-sided figure whose opposite sides are parallel.

With the dividers compare the lengths of **KL** and **NM**; also the lengths of **KN** and **LM**. With the bevel compare the magnitudes of the angles **NKL** and **NML**; also the magnitudes of the angles **KLM** and **KNM**.

Construct two or more parallelograms with sides of different lengths, and angles of different magnitudes; and in the case of each compare the magnitudes of the opposite sides and angles.

The result of such observations will be that **the opposite sides and angles of parallelograms are equal**.

We may also *prove* this as follows: Draw **KM**, the **diameter** or **diagonal**, as it is called, of the parallelogram. We have in the two triangles **NKM**, **LMK**,—

The side **KM** common to both,
the alternate angles **NKM**, **LMK** equal,
“ “ “ **NMK**, **LKM** “

Hence (Ch. III., 5) these triangles are equal in all respects, *i.e.*,

$$KN = ML,$$

$$KL = MN,$$

$$\angle KNM = \angle KLM.$$

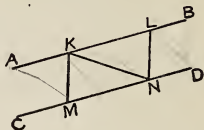
$$\text{Also } \angle NKM = \angle LMK,$$

$$\text{and } \angle LKM = \angle NMK;$$

therefore adding $\angle NKL = \angle NML$.

Of course the triangle **KNM**, if cut out, can be fitted on the triangle **MLK**, and is equal to it, *i.e.*, **the diagonal of a parallelogram bisects it**. ✓

2. Draw a pair of parallel lines, **AB** and **CD**. In **AB** take any length **KL**, and, adjusting the dividers to it, in **CD** mark off an equal length **MN**. Join **K, M** and **L, N**.



Using the dividers, what do you note with reference to the lengths of **KM** and **LN**? Using the bevel or parallel rulers, what do you note with reference to the direction of **KM** and **LN** with respect to one another?

Draw other parallel lines, mark off on them equal lengths, join the extremities of these equal lengths, and repeat the examination as to the lengths and relative direction of the joining lines.

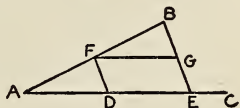
The result of such observations will be that **the straight lines joining the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel**.

We may *prove* this as follows: In the triangles **LKN, MNK**, the sides **LK, KN** are equal to the sides **MN, NK**; and the angles **LKN, MNK** are equal. Hence

these triangles are equal in all respects., Therefore **LN** and **MK** are equal. Also the alternate angles **LNK** and **MKN** are equal, and therefore **LN** and **MK** are parallel.

3. With the power of drawing parallel lines we have another means of bisecting a line, indeed of dividing a line into any number of equal parts:

Let **AB** be the line to be bisected. Draw through **A** any other line **AC**, and with the dividers mark off on it equal lengths **AD**, **DE**. Join **BE**, and with the parallel rulers draw **DF** parallel to **BE**. **F** is the bisection of **AB**. For, drawing **FG** parallel to **AC**, the triangles **ADF**, **FGB** are evidently equal, and **AF** is equal to **FB**.

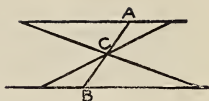


In employing this method of bisecting a line, we may avoid altogether drawing the lines **AC**, &c. For, placing the edge of the ruler against **A**, and, close to the edge of the ruler, with the sharp points of the dividers, mark the points **D** and **E** (the distances **AD**, **DE** being equal). Then place the edge of the parallel rulers against the points **B** and **E**, and move the edge, parallel to itself, back to **D**. The point in which the edge cuts **AB** is its middle point, and can be marked with a point of the dividers.

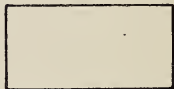
It is well to so place **AC** and the points **D** and **E**, that the lines **DF**, **EB** cut **AB** at nearly 90° . The point **F** is thus located with most definiteness.

We leave to the pupil to discover for himself, following the suggestion here given, a means of dividing a straight line into any number of equal parts.

4. Having drawn two parallels, adjust the points of the dividers to a distance of, say, 1 inch from one another. Place one point at **A**, and let the other point of the dividers meet the other parallel at **B**. Then $\frac{1}{2}$ inch from **A** gives the middle point of **AB**. Draw a number of lines through **C**, and terminated by the parallels. Using the dividers, **C** will be found to be the middle point of all these lines.



5. If the angles of a parallelogram are right angles, it is called a **rectangle**.



If the adjacent sides, and therefore all the sides, of a parallelogram are equal, it is called a **rhombus**.



If the angles of the rhombus are right angles, the figure is called a **square**, *i.e.*, a square is a four-sided figure with all its sides equal, and all its angles right angles.



Construct the following parallelograms :

Sides 2 inches and $3\frac{1}{2}$ inches, and included angle 45° .

Sides 2 and 3 inches, and included angle 58° .

.

In all cases test the equality of the opposite sides and angles.

Construct the following rhombuses :

Sides 3 inches, and one angle 60° .

Sides 5 inches, and one angle 75° .

Sides $3\frac{1}{2}$ inches, and one angle 15° .

.

Construct the square whose side is 2 inches ;
whose side is $4\frac{1}{2}$ inches ; whose side is 3 inches ;

.

Summary.

1. Def. A *parallelogram* is a four-sided figure whose opposite sides are parallel. Every parallelogram has its opposite sides equal, also its opposite angles, and each diagonal divides it into two triangles which are equal in all respects.

2. The straight lines joining the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel.

3. To bisect a given straight line.

4. To divide a given straight line into any number of equal parts.

Exercises.

1. Define a parallelogram. What other facts are true about it? What sort of figure does it become if :

(1) All its sides are equal?

(2) One of its angles is 90° ?

(3) All its sides are equal and one angle 90° ?

2. With two equal triangles, cut out of paper, form a parallelogram.

3. Draw both diagonals in a number of parallelograms, and examine how the point in which the diagonals intersect divides them. Give proof.

4. With compasses and ruler only, construct a four-sided figure with opposite sides equal. How are opposite sides placed with respect to each other? Apply tests. Give proof.

5. With protractor and ruler, construct a four-sided figure with one pair of opposite angles equal and each 75° , and the other pair of opposite angles equal and each 105° . What is the figure? Apply tests. Give proof.

6. A parallelogram has its diagonals equal. What sort of figure must it be?

7. Draw a line 3 inches long and divide it into 7 equal parts.

8. Draw a rhombus. Show that its diagonals are perpendicular to each other.

9. Describe a circle, and drawing any two diameters, join their extremities. What is the figure so formed?

10. What sort of figure would be formed in the preceding question, if the diameters were drawn at right angles to each other?

11. Draw a rectangle, a square, a rhombus, and a parallelogram whose adjacent sides are unequal. Ascertain in which of these figures

(1) The diagonals bisect each other.

(2) The diagonals are perpendicular to each other.

(3) The diagonals are equal.

(4) Each diagonal divides the figure into two equal triangles.

(5) If the figure be folded about its diagonal the parts fit exactly on each other.

12. The sides of a parallelogram are $2\frac{1}{2}$ and $1\frac{1}{2}$ inches, and its diagonal is $1\frac{3}{4}$ inches. Construct it.

13. On a line of 1 inch as diagonal, construct a rhombus whose sides are 2 inches long.

14. On a line of 1 inch as diagonal, construct a square. Measure its sides.

CHAPTER VIII.

Certain Relations in Area between Parallelograms and Triangles.

1. Draw a rectangle $ABCD$, making AB 4 inches long and AD 3 inches long.

From each inch division on AB draw lines parallel to AD and BC , and from each inch division on AD draw lines parallel to AB and DC .



The figure will thus be divided into 12 squares, each being 1 square inch in area.

That is, area of $ABCD = 12$ sq. inches.
 $= 4 \times 3$ sq. inches
 $= \text{length} \times \text{breadth}.$

Repeat the same experiment with rectangles of different dimensions, proving in each case that

area of rectangle = length $\times$ breadth.

2. Draw any parallelogram $ABCD$ and cut it in two



FIG. 1.

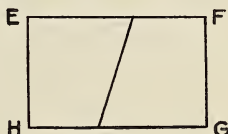


FIG. 2.

along a line perpendicular to the parallel sides (see dotted line Fig. 1). Fit the two pieces together again as shown in Fig. 2. The resulting figure **EFGH** will be seen to be a rectangle whose length **HG** = the base **DC** of the parallelogram, and whose width **EH** = the perpendicular width of the parallelogram.

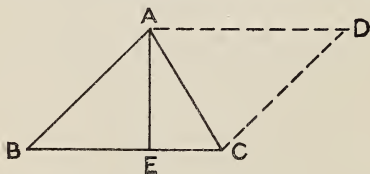
Now, area of **ABCD** = area of **EFGH**
 $\quad = \mathbf{HG \times EH}$ (Section 1)
 $\quad = \text{length of parallelogram} \times \text{its perpendicular width.}$

$\therefore$ The area of a parallelogram = the area of a rectangle on the same base and of the same height or altitude.

Or, the area of a parallelogram = its length $\times$ its perpendicular width.

3. From Section 2 it follows that if two parallelograms have the same or equal bases, and are of the same altitudes, they are equal in area.

4. Draw any triangle **ABC**. Draw **AE** its altitude.



Through **C** draw a line parallel to **AB**.

Through **A** draw a line parallel to **BC**.

Let these lines meet in **D**.

Then **ABCD** is a parallelogram (definition).

$\therefore$ by Chapter VII., Sec. 1,

Area of triangle $ABC = \frac{1}{2}$ area of parallelogram $ABCD$.

Now, area of $ABCD = BC \times AE$ (AE being altitude of the parallelogram $ABCD$).

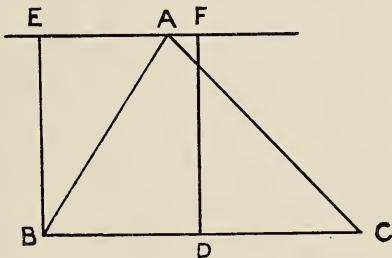
$\therefore$ Area of triangle $ABC = \frac{1}{2} (BC \times AE)$.

$\therefore$ **Area of any triangle** $= \frac{1}{2}$ base $\times$ altitude.

5. It follows from Section 4 that **triangles on the same or equal bases, and of the same altitude, are equal in area.**

6. To construct a rectangle equal in area to a given triangle.

Let ABC be the given triangle. Draw EAF through A , parallel to BC . Bisect BC in D . Through B and D draw lines BE and DF perpendicular to BC , terminated by EAF . Then $EBDF$ is the required rectangle.



It is a rectangle by construction, also:

Its area $= BD \times FD$

$= \frac{1}{2} BC \times$ altitude of triangle ABC

$=$ area of triangle ABC . (Sec. 5.)

Hence to construct a rectangle equal to a triangle, bisect the base of the triangle, and on the half-base construct a rectangle of the same altitude as the triangle.

Can you suggest another method of performing this construction?

Construct rectangles equal in area to the following triangles:

Sides 3, $3\frac{1}{2}$, 4 inches.

Sides 3, 4 inches, and included angle 50° .

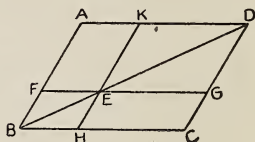
Base 3 inches, and angles at base 45° and 75° .

.

Calculate approximately, in square inches, the areas of the triangles in 6 by finding the lengths of the sides of the rectangles which have been constructed equal to the triangles.

It will be interesting for the teacher to calculate the areas of such triangles by the usual trigonometrical formulæ, that he may inform the class as to the closeness of their approximations reached by instrumental methods.

7. In the adjoining figure **BD** is the diagonal of each of the parallelograms **ABCD**, **FBHE**, **KEGD**, and therefore bisects each of them. Hence we have



triangle **ABD** = triangle **CBD**,

“ **FBE** = “ **HBE**,

“ **KED** = “ **GED**.

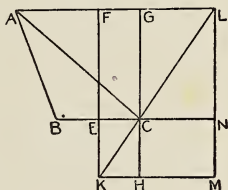
Therefore the parallelogram **AFEK** is equal to the parallelogram **CHGE**. Note the position of these parallelograms with respect to the diagonal **BD**.

Note—In the above figure **KEGD** and **FEHB** are spoken of as the parallelograms about the diagonal **BD**. **AKEF** and **EGCH** are described as the complements of the parallelograms about the diagonal **BD**, because they, with the parallelograms about the diagonal, complete the whole figure.

The complements of the parallelograms about the diagonal of a parallelogram are equal in area.

8. In constructing a rectangle equal to a given triangle (Ch. VIII., 6), one of the sides of the rectangle is half the base of the triangle. We may, however, construct a rectangle equal to any triangle, and give to one of the sides of the rectangle any length we choose.

Thus having constructed **FECG** equal to the triangle **ABC**, suppose we wish to make a rectangle equal to the triangle, with one of its sides of length **CH**. Complete the rectangle **EKHC**, and let **KC**, **FG** meet in **L**. Draw the remain-



ing lines as indicated in the figure. Then the rectangle **CM**, whose side **CH** is of the required length, is equal to the rectangle **FC** (Ch. VIII., 7), which is equal to the original triangle **ABC**.

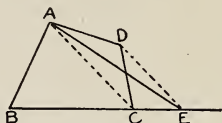
Construct rectangles, each with a side of 2 inches, equal to the triangles in 6.

The sides of a triangle are 2, 3 and 4 inches. Construct a rectangle equal to it, having one side of $2\frac{1}{2}$ inches. Measuring the other side of the rectangle, calculate approximately the area of the rectangle, *i.e.*, of the triangle.

The sides of a triangle are 3 and 4 inches, and the included angle is 50° ; construct a rectangle equal to it, one of whose sides is 2 inches. Measuring the other side of the rectangle to the nearest sixteenth of an inch, calculate approximately the area of the rectangle, *i.e.*, of the triangle.

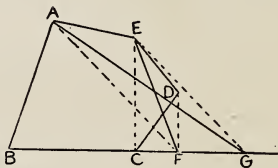
9. If we wish to construct a rectangle equal in area to a polygon, and thence, if necessary, calculate the area of the polygon, it is well first to construct a triangle equal to the polygon by the following method :

Let $ABCD$ be a quadrilateral whose area we wish to calculate. Place the edge of the parallel rulers along AC , and slide one bar out until the edge reaches D , and mark the point E in BC produced. AC is parallel to DE , and therefore the triangle ACE is equal to the triangle ACD (Ch. VIII., 5); and therefore the triangle ABE is equal to the quadrilateral $ABCD$.



We may then measure the perpendicular height of ABE and its base BE : their product divided by 2 gives the area of ABE (Ch. VIII., 4), and therefore of $ABCD$.

Suppose we wish to find the area of the pentagon $ABCDE$. Place the edge of the parallel rulers along CE , and slide one bar out until the edge reaches D , and mark the point F in BC produced. DF is parallel to CE , and therefore the triangle ECF is equal to the triangle ECD . Thus the quadrilateral $ABFE$ is equal to the pentagon $ABCDE$.



Again, place the edge of the parallel rulers along AF , and slide one bar out until the edge reaches E , and mark the point G in BC produced. EG is parallel to AF , and therefore the triangle GAF is equal to the triangle EAF ; and therefore the triangle ABG is equal to the quadrilateral $ABFE$, and to the pentagon $ABCDE$.

We may then measure the perpendicular height of **ABG** and its base **BG**: their product divided by 2 gives the area of **ABG** and therefore of **ABCDE**.

In the preceding figures the dotted lines need not be drawn. The point **E** in the former figure, and the points **F** and **G** in the latter, are where the edge of the parallel rulers cuts **BC**.

Had we selected **AB** as our base, instead of **BC**, our resulting triangle would have had a different height and base, but would necessarily have been of the same area as **ABE** or **ABG**.

The sides **AB**, **BC**, **CD** of a quadrilateral are 3, $2\frac{1}{2}$ and 2 inches; the angles **ABC**, **BCD** are 70° and 60° . Construct a triangle equal to it in area, and thence calculate its area. Here use **BC** and next **AB** as bases; and, by comparing the areas of the resulting triangles, obtain a test of the accuracy of your construction.

Construct several quadrilaterals and pentagons, and find triangles equal to them in area. In each case construct the triangle in two different ways (as in the preceding example) and, by comparing the areas of such triangles, obtain a test of the accuracy of your construction.

In all cases where numerical measurements are made, such measurements are necessarily approximate, and therefore in examples such as the preceding the areas will be found only approximately. Hence where a numerical area has been reached by two different ways, we are to expect only approximate agreement.

Chapters XVIII., XIX., and XX., relating to similar triangles, may now be taken up if thought desirable.

Summary.

1. The area of a rectangle = length $\times$ breadth.
2. The area of a parallelogram = length $\times$ its perpendicular width.
3. Parallelograms on the same base, or on equal bases, and of the same altitudes are equal in area.
4. The area of triangle = $\frac{\text{base} \times \text{altitude}}{2}$.
5. Triangles on the same base, or on equal bases, and of the same altitudes are equal in area.
6. Two methods of constructing a rectangle equal to a given triangle.
7. The complements of the parallelograms about the diagonals of a parallelogram are equal in area.
8. On a line of given length construct a rectangle equal in area to a given triangle.
9. To reduce a figure of any number of sides to a triangle, and hence to find its area.

Exercises.

1. If two triangles have the same base and equal areas, what relation exists between their altitudes?

If their vertices be joined, what direction does the joining line take with respect to the common base?

2. If two triangles have equal bases and equal areas, what relation exists between their altitudes?

If the bases be in the same straight line and the vertices be joined, what direction does the joining line take with respect to the line in which the bases lie?

3. If ABC be a triangle, and D be the middle point of AB, what relation exists between the areas of the triangle CDA and CDB?

4. If D and E be the middle points of the sides AB, AC of the triangle ABC, what relation exists between the areas of the triangles DBC, EBC? What do you infer as to the position of DE with respect to BC?

5. Construct a quadrilateral, and bisect the sides. What positions do the lines joining the bisections of adjacent sides occupy with respect to the diagonals?

What is the figure formed by joining in succession the points of bisection?

How do the lines joining the bisections of opposite sides of the quadrilateral divide each other? Give reason.

6. Two sides of a quadrilateral are parallel and of lengths $2\frac{1}{2}$ and 3 inches. The distance of these sides apart is $\frac{3}{4}$ of an inch. What is the area of the quadrilateral? (Join two opposite corners, and find area of each triangle.)

7. The sides of a rectangle are 2 and 3 inches. Find by geometrical construction a rectangle equal to it in area, one of whose sides is $2\frac{1}{2}$ inches. Test by measurement and numerical calculation the accuracy of your construction.

8. The sides of a triangle are 3, $4\frac{1}{2}$ and 5 inches. Construct a rectangle equal to it in area with one side $2\frac{1}{2}$ inches. Construct also a rectangle equal to it in area, one of whose sides is 3 inches.

9. In the previous question, how could you construct on the 3-inch line a parallelogram, with one of its angles 45° , and equal to the given triangle?

10. On a base of $1\frac{1}{2}$ inches construct two parallelograms of equal area, one having a side of $2\frac{1}{4}$ inches and an angle of 75° , and the other an angle of 120° .

11. In a quadrilateral ABCD, $AB=1\frac{1}{2}$, $BC=1\frac{3}{4}$, $CD=2\frac{1}{4}$ inches; $\angle ABC=60^\circ$, $\angle BCD=75^\circ$. Construct a triangle and also a rectangle equal to it in area. Hence calculate its area, approximately, in square inches.

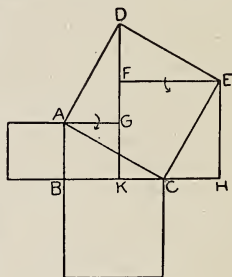
12. A quadrilateral ABCD has AB (2 in.) and CD ($3\frac{1}{2}$ in.) parallel, and $1\frac{1}{2}$ in. apart. Construct a rectangle equal to it in area, one of whose sides is $1\frac{1}{2}$ in.

13. ABC is a triangle, and D, E the middle points of AB, AC. BE, CD intersect in O. Join AO, and show that the triangles OAB, OBC, OCA are equal in area.

CHAPTER IX.

Squares on Sides of a Right-angled Triangle.

1. Let the angle **B** of the triangle **ABC** be 90° . Describe squares on the sides of **ABC**, as in the figure. Draw the lines **AG**, **EF**, **CH** parallel to **BC**; and the lines **DK**, **EH** parallel to **AB**.



Then measurement (with dividers for lines, and bevel for angles) will show that the triangle **AGD** is in all respects equal to the triangle **ABC**; and cutting out the triangle **AGD**, it may be turned about **A**, in the direction indicated by the arrow head, into the position **ABC**. Measurement will also show that the triangle **EFD** is in all respects equal to the triangle **EHC**; and cutting out the triangle **EFD**, it may be turned about **E**, in the direction indicated by the arrow head, into the position **EHC**. We thus have the square on **AC** converted into **ABKG** and **FKHE**, which will be found to be the squares on **AB** and **BC**.

Repeat the same construction, measurements, and superposition in the case of the following triangles:

AB = $1\frac{3}{4}$ inches, **BC** = $2\frac{1}{2}$ inches; **ABC** = 90° .

AB = $1\frac{1}{2}$ inches, **BC** = $2\frac{1}{2}$ inches; **ABC** = 90° .

AB = 2 inches; **ABC** = 90° , **BAC** = 60° .

The result of these observations may be stated thus:

In any right-angled triangle the square which is described on the side subtending the right angle, is equal to the sum of the squares described on the sides containing the right angle.

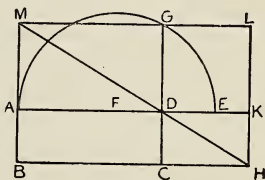
Two sides of a right-angled triangle about the right angle, are 3 and 4. What is the length of the third side?

If a string or rope of length 12 be broken into lengths 3, 4 and 5, and these be formed into a triangle, such triangle is right-angled.

If the lengths of the pieces of rope be 30, 40 and 50, the triangle formed with them is also right-angled.

2. A square may be constructed equal in area to any rectangle, as follows:

Let $ABCD$ be the rectangle. Make DE equal to DC , and find F the middle point of AE . Describe the semicircle, and produce CD to G . Then the square on DG is equal to the rectangle $ABCD$.



For, describe the square $DGLK$ on DG , and let LK and BC meet in H . Then, if the figure has been accurately constructed, on producing the lines LG , HD and BA , they will be found to all pass through one point, M . Hence (Ch. VIII., 7) the square $GDKL$ is equal to the rectangle $ABCD$.

In the succeeding constructions it is of course absolutely necessary that the three lines corresponding to LG , HD , and CA pass through the same point (M).

Describe the rectangle whose sides are 2 and 3 inches. Construct, as above, the square equal to it. Measure in inches and fractions of an inch the side of the square, and thence verify the accuracy of your construction.

Proceed similarly with the rectangle whose sides are 1 and 4 inches.

Also with the rectangle whose sides are 9 and 16 sixteenths of an inch.

Also with the rectangle whose sides are 18 and 32 sixteenths of an inch. The sides of this rectangle are twice those of the former: note the number of times its area is greater than that of the former; note the same with respect to the resulting squares.

Summary.

1. In any right-angled triangle the square on the side opposite the right angle is equal to the sum of the squares on the other two sides.

2. If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the triangle is right-angled.

3. To describe a square equal in area to a given rectangle.

Exercises.

1. Describe two squares and construct a third equal to their sum.

2. Describe a square which shall contain 13 square inches.

3. The side of an equilateral triangle is 2 inches. What is the length of the perpendicular from any angle on the opposite side?

4. Draw two lines CA, CB at right angles to each other and each of length 1 inch. What is the area of the square on AB?

5. In the preceding question, if CB is 2 inches long, what is the length of AB?

6. Draw a straight line whose length will be $\sqrt{10}$ inches.

7. In question 4, draw AD (=1 in.) perpendicular to AB. What is the area of the square on DB?

8. Draw a straight line whose length will be $\sqrt{3}$ inches.

9. Describe a triangle such that the square on one side is greater than the sum of the squares on the other two sides, say with sides of 2, 3 and 4 inches. What relation does the angle opposite the greatest side bear to a right angle? Measure it with protractor.

10. Describe a triangle such that the square on one side is less than the sum of the squares on the two other sides, say with sides of $1\frac{1}{2}$, $2\frac{1}{2}$ and $2\frac{3}{4}$ inches. What relation does the angle opposite the one side bear to a right angle?

11. Construct any quadrilateral with its diagonals at right angles to each other. Show that the sum of the squares on two opposite sides is equal to the sum of the squares on the other two sides.

12. ABC, DEF are right-angled triangles with the hypotenuses AB, DE equal, and also the sides BC, EF equal. Show that the remaining sides AC, DF are equal.

13. The sides of a triangle are $1\frac{1}{2}$, 2, $2\frac{1}{2}$ inches. Construct a rectangle equal to this triangle. Construct a square equal to the rectangle so formed.

14. The side of an equilateral triangle is 2 inches. Construct a square equal to it.

15. If a right-angled triangle have one of the acute angles double the other, divide it into two triangles, one equilateral and the other isosceles.

16. Bisect the hypotenuse of a right-angled triangle. What is the relation between the distances of the point so obtained from the three angles?

17. ABC is a right-angled triangle, and CD is drawn perpendicular to the hypotenuse. Examine the relations between the angles of the three triangles ABC, ACD, BCD. Give reasons.

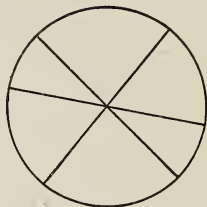
18. In the preceding question, show by experiment (1) that the rectangle contained by AD, DB is equal to the square on CD; (2) that the rectangle contained by AB, AD is equal to the square on AC; (3) that the rectangle contained by AB, BD is equal to the square on BC.

CHAPTER X.

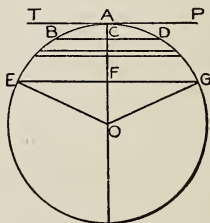
The Circle. Its Symmetry. Tangents. Finding of Centre.

1. The fundamental quality of the circle, next to the equality of its radii, is its **symmetry**.

In the first place, every line drawn through the centre from circumference to circumference (*i.e.*, every diameter) is bisected at the centre. This is called **central symmetry**.



In the second place, every chord drawn at right angles to a diameter is bisected by that diameter. This is called **axial symmetry**. Thus the chord **EFG** being perpendicular to **OA**, the parts **EF**, **FG** are equal. Measurement will establish the equality of these parts. Or we may *prove* it thus:



The rt. $\angle$ ^{les} at **F** are equal.

Because **OE = OG**, $\therefore \angle OEF = \angle OGF$.

Hence $\angle$ ^{les} at **O** are equal.

Also sides **EO**, **OF** = sides **GO**, **OF**;

$\therefore$ (Ch. III., 2) **EF = FG**.

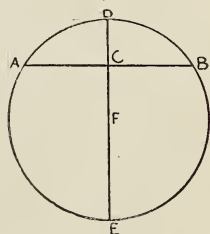
And hence all chords perpendicular to a diameter are bisected by it.

2. As the chord **BCD** moves parallel to itself down to **A**, since the parts on each side of the diameter are always equal, when one part vanishes, the other vanishes also. Thus the line **TAP**, through **A** parallel to **BCD**, while it **touches** the circle, does not cut it. Such a line (**TAP**) is called the **tangent** to the circle at **A**. That is to say, **a tangent is a line drawn through the extremity of a diameter, and at right angles to it.**

The tangent is evidently a straight line which meets the circle, but does not cut it: this is sometimes given as the definition of a tangent.

3. Since a diameter bisects every chord to which it is at right angles, therefore a line drawn through the bisection of a chord and at right angles to it, must be a diameter. Hence if the centre of any circle be not indicated, we may reach it by the following construction:

Draw any chord **AB**. Bisect it at **C**. Draw **DCE** perpendicular to **AB**. **DE** must pass through the centre. Hence, bisecting **DE** at **F**, **F** must be the centre of the circle.



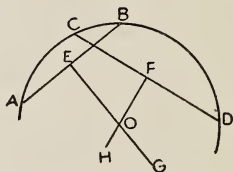
We may describe circles without marking their centres by placing a piece of thin wood or cardboard under the stationary point of the compasses, removing this piece of wood or cardboard when the circle is described.

Circles being thus described, or being obtained by marking with the pencil about a round object placed on the paper (coin, bottom of ink bottle, plate, &c.), attempts should be made to locate the centre by the eye's judgment. We may afterwards test the correct-

ness of this by making the preceding construction, and finally test the accuracy of the construction by trying with such centre to reproduce the circle by using the compasses.

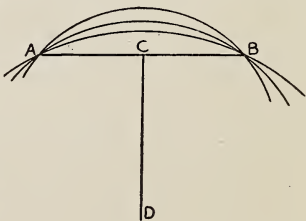
It will be found, of course, that the greater the circle, the greater will be the difficulty of locating, with the eye's judgment, the position of the centre. The same difficulty occurs in locating the bisection of a straight line with the eye.

4. If only an arc of the circle be given, we may find the centre, and complete the circle, as follows:



Draw two chords **AB** and **CD**; find their middle points **E** and **F**; through these middle points draw perpendiculars **EG** and **FH**. The centre of the circle must lie on each of the lines **EG** and **FH** (Ch. X., 3), and therefore must be at **O**.

Arcs of circles should be described without marking the centres, by the method suggested in § 3. The positions of the centres should then be judged with the eye; afterwards constructed for, and the accuracy of the construction tested by attempting, with the compasses, to describe the arc with the centre so obtained.



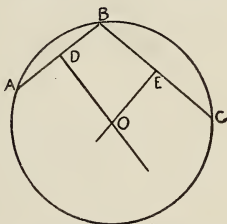
5. If any line **AB** be bisected at **C**, and **CD** be drawn perpendicular to it, then all points in **CD** are equally distant from **A** and **B**. Hence, if we place the sharp

point of the compasses at *any* point on **CD**, and the pencil end at **A**, and describe a circle, it will also pass through **B**. We thus get an unlimited number of circles through **A** and **B**, all of which have their centres at different points on **CD**.

Draw a line **AB** of 2 inches, and describe circles passing through **A** and **B**, with radii 1, 2, 3 and 4 inches.

6. We can readily obtain a method for describing a circle to pass through any three points:

Let **A**, **B**, **C** be the three points. Draw **DO** from the middle point of **AB** at right angles to it; and draw **EO** from the middle point of **BC** at right angles to it. Then all points in **DO** are equally distant from **A** and **B**; and all points in **EO** are equally distant from **B** and **C**. Hence **O** is equally distant from **A**, **B** and **C**; and placing the sharp point of the compasses at **O** and the pencil end at **A**, and describing a circle, it will pass through **B** and **C**, if the construction has been accurate.



AB is 1 inch, **BC** is 2 inches, and angle **ABC** is 120° . Describe a circle to pass through **A**, **B** and **C**.

AB is $1\frac{1}{2}$ and **BC** $2\frac{1}{2}$ inches, and the angle **ABC** is 75° . Describe a circle to pass through **A**, **B** and **C**.

AB is $1\frac{1}{2}$ and **BC** $2\frac{1}{2}$ inches, and the angle **ABC** is 90° . Describe a circle to pass through **A**, **B** and **C**. Show that its centre bisects **AC**.

Mark sets of three points in various positions with respect to one another, and describe a circle to pass through each set.

Summary.

1. A circle possesses central symmetry, *i.e.*, every diameter is bisected at the centre.
2. A circle possesses axial symmetry, *i.e.*, every chord drawn at right angles to a diameter is bisected by that diameter.
3. A tangent is a straight line which meets the circle, but, being produced, does not cut it. Any line drawn at right angles to the diameter of a circle at either extremity is a tangent to the circle.
4. Given a circle, to find its centre.
5. Given an arc of a circle, find the centre of this circle and complete it.
6. Describe a circle passing through three given points.

Exercises.

1. Describe a circle; draw in it any chord; join the centre to the extremities of the chord; and drop a perpendicular from centre on chord.

What is the relation between the angles the radii make with the chord? What between the angles the radii make with the perpendicular? What between the segments of the chord made by the foot of the perpendicular?

2. With the bevel or protractor construct two equal angles at the centre of a circle, and draw the chords which subtend these angles. What is the relation between these chords? Apply test. Give reasons.

3. Describe a circle, and with dividers and ruler place two equal chords in it. Join the ends of the chords to the centre. What is the relation between the angles these equal chords subtend at the centre? Apply test. Give reasons.

4. As in the previous question, in a circle place two equal chords, and from the centre drop perpendiculars on them. What is the relation between these perpendiculars? Apply test. Give reasons.

5. Describe a circle of radius 3 inches, from the centre draw two equal lines of length 2 inches, and through the extremity of each draw a line at right angles to it, so obtaining two chords at equal distances from the centre. What is the relation between the lengths of these chords? Apply test. Give reasons.

6. Repeat the construction of question 4, making the chords unequal. What is now the relation between these perpendiculars?

7. In question 5 make one of the lines drawn from the centre one inch. What is now the relation between the lengths of the chords?

8. Summarize the facts obtained in answering questions 4, 5, 6, 7.

9. The sides of a triangle are $2\frac{1}{2}$, 3 and $3\frac{1}{2}$ inches. Describe a circle passing through the angular points.

10. The sides of a triangle are 2, 3 and 4 inches. Describe a circle passing through the angular points.

11. The sides of a triangle are 3, 4 and 5 inches. Describe a circle passing through the angular points.

12. Two chords of a circle with one end of each common, are of lengths 2 and 3 inches, and make an angle of 60° with each other. Describe the circle.

13. The sides of a rectangle are $1\frac{1}{2}$ and $2\frac{1}{2}$ inches. Describe a circle passing through all the angular points.

14. Describe a parallelogram ABCD, not being a rectangle. Can a circle be described passing through its angular points? (Every circle through A and B has its centre on the line which bisects AB at right angles.)

15. The diameter of a circle is 30 inches, and a chord is 24 inches. How far is the chord from the centre?

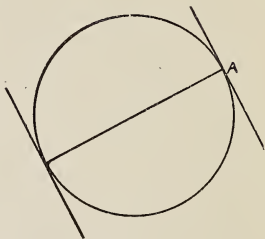
16. The radius of a circle is $3\frac{1}{4}$ inches. What is the length of a chord whose distance from the centre is $1\frac{1}{4}$ inches?

17. Construct two intersecting circles, join their centres, and through either of the points of intersection, draw a line parallel to the line joining centres, and terminated by the circumferences. What relation in length between the second line drawn and the line joining the centres? Apply test. Give reasons.

CHAPTER XI.

Tangents to Circles, and Circles Touching One Another.

1. To draw the tangent at any point **A** on the circumference of a circle, draw the diameter through **A**, and draw at **A** the perpendicular to this diameter. The perpendicular is a tangent to the circle (Ch. X., 2).



Evidently the tangents at opposite ends of a diameter are parallel to one another.

Construct a circle of radius $2\frac{1}{2}$ inches. Draw radii at intervals of 30° , and draw the tangents at the ends of these radii, producing each both ways until it meets the adjacent tangents.

Construct a circle of radius 2 inches. Draw radii at intervals of 45° , and draw tangents at the ends of these radii, producing each both ways until it meets the adjacent tangents.

Construct a circle of radius $1\frac{5}{8}$ in. Draw radii at intervals of 72° , and draw tangents at the ends of these radii, producing each both ways until it meets the adjacent tangents.

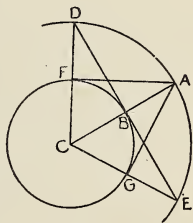
In each of the three preceding constructions, the resulting figure about the circle should have equal sides and equal angles. The equality of the sides

(measured with the dividers) and the equality of the angles (measured with the bevel) may be regarded as a test of the accuracy of the construction.

Any two diameters in a circle are drawn, inclined at an angle of, say, 30° to each other, and tangents at the ends of these diameters are constructed. What quadrilateral figure about the circle do the tangents form? Measure its sides.

2. From a point without a circle, evidently two tangents can be drawn to the circle. To draw those from **A** to the circle **FBG**:

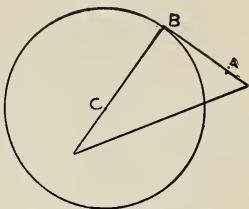
Join **AC**, cutting the circle in **B**. Describe a second circle **DAE**, with centre **C** and radius **CA**. Draw **DBE** perpendicular to **CB**. Join **CD** and **CE**, cutting the small circle in **F** and **G**. Then **AF** and **AG** are the tangents from **A**.



For, the triangles **ACF** and **DCB** are equal. But the angle **CBD** is a right angle; therefore the angle **CFA** is a right angle, and **AF** is a tangent to the circle (Ch. X., 2). In the same way we may prove that **AG** is a tangent.

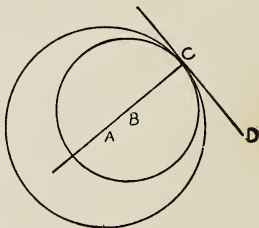
Symmetry suggests that the tangents **AF**, **AG** are equal in length, and that they make equal angles with **AC**. The truth of this may be tested by measurement. It may also be *proved* as follows: Because **CDE** is an isosceles triangle, and the angles at **B** right angles, therefore the triangles **CDB**, **CEB** are equal in all respects. But the triangle **CAF** is equal in all respects to **CDB**; and the triangle **CAG** is equal in all respects to the triangle **CEB**. Therefore the triangles **CAF** and **CAG** are equal in all respects. Hence **AF**, **AG** are equal, and the angles at **A** are equal.

In practice, an easy way to draw a tangent from any point **A**, outside the circle, is as follows: Place the set-square so that one of its sides passes through **A** and the other through **C**, the centre of the circle. Then so adjust the instrument that the right angle rests on the circumference at, say, **B**. **AB**, a tangent through **A**, may then be drawn.

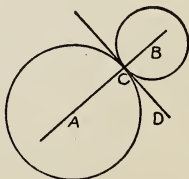


Construct a circle of radius $1\frac{1}{2}$ in., and draw any line through its centre. From points on this line at distances from the centre 2, $2\frac{1}{2}$, 3 in., draw tangents to the circle.

3. Let a circle be described with centre **A**, and the tangent at any point **C** be drawn; and let, with centre **B**, on **AC**, and radius **BC**, another circle be drawn. Then both circles have **CD** for tangent. Both *touching* the same line at the same point, they are said to *touch* one another,—in this case *internally*.



Let a circle be described with centre **A**, and the tangent at any point **C** be drawn; and let, with centre **B**, in **AC** produced, and radius **BC**, another circle be described. Then both circles have **CD** for tangent. Both *touching* the same line at the same point, they are said to *touch* one another,—in this case *externally*.



Evidently, whether circles touch internally or externally, the straight line joining their centres passes through the point of contact.

Describe circles of radii $1\frac{1}{2}$ and 2 inches to touch (1) externally, (2) internally.

Construct a series of circles of radii $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{2}$, $\frac{3}{8}$ inches, their centres being in the same straight line, and each circle touching the preceding (and succeeding) externally.

Describe circles of radii as in preceding, but each circle touching the others at the same point, internally.

Two circles of radii 1 and $1\frac{1}{2}$ inches touch one another externally. Describe a circle of radius $\frac{3}{4}$ inch, to touch both of them externally. (This involves the construction of a triangle with sides $2\frac{1}{2}$, $2\frac{1}{4}$ and $1\frac{3}{4}$ inches.)

Make the same construction as in the preceding question, when the first two circles have radii 1 and $1\frac{3}{4}$, and the third a radius of $\frac{3}{4}$ inch.

The sides of a triangle are 3, $2\frac{1}{2}$ and $1\frac{3}{4}$ inches. With the angular points of this triangle as centres, describe three circles with radii $1\frac{1}{8}$, $\frac{5}{8}$, $1\frac{7}{8}$ inches, so that each may touch the other two.

When the sides of a triangle are 4, 3 and $2\frac{1}{2}$ inches discover the circles whose radii are such that in like manner each will touch the other two, the angular points of the triangle being centres of the circles.

Summary.

1. To draw a tangent to a circle from any point on its circumference.
2. To draw a tangent to a circle from any point outside it.
3. To draw any number of circles touching each other internally.
4. To draw any number of circles touching each other externally.

5. If two or more circles touch each other, whether externally or internally, the line joining their centres passes through the point of contact.

6. Tangents drawn to a circle from the same external point are equal.

Exercises.

1. Describe a circle of radius $1\frac{1}{4}$ in.; draw any line in plane of paper; draw a tangent parallel to this line. (From centre drop perpendicular on line, and at point of intersection with circle draw tangent.)

2. Draw any line and draw circles of radii 1, $1\frac{1}{2}$ and 2 inches, touching the line at any points.

3. Describe two circles of radii 1 inch and $2\frac{1}{4}$ inches, so as to touch any line at points 3 inches apart. Do the circles touch one another?

4. A tangent of length 4 inches is drawn from a point to a circle of radius 3 inches. How far is the point from the centre of the circle?

5. A tangent is drawn to a circle of radius 1 inch, and another circle, concentric with the former, is described of radius 2 inches. What is the length of the tangent between the point where it is intercepted by the second circle and the point of contact? What angle does the intercepted portion of the tangent subtend at the common centre?

6. Describe two circles of radii 1 and $1\frac{1}{2}$ inches, to touch one another internally; and describe a circle of radius $2\frac{1}{2}$ inches to touch both, and contain both.

7. Describe three circles of radii $2\frac{1}{4}$, 3 and $3\frac{1}{4}$ inches, so that each may touch the other two.

8. Describe two concentric circles of radii 1 and 3 inches, and describe a number of circles touching both of them.

9. Two circles touch internally at A, and ABC is drawn to meet the circles at B and C. What is the position of radii to B and C with respect to each other? Apply test. Give reasons.

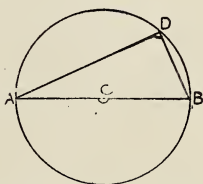
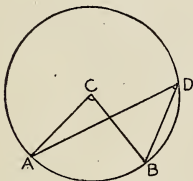
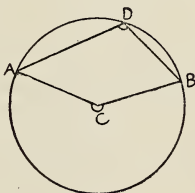
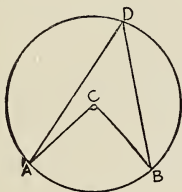
10. Two circles touch externally at A, and ABC is drawn to meet the circles at B and C. What is the position of radii to B and C with respect to each other? Apply test. Give reasons.

11. Draw two tangents to a circle from an external point, and join the points of contact. What is the relation between the angles this "chord of contact" makes with the tangents? Apply test. Give reasons.

CHAPTER XII.

Angles in a Circle.

1. The angles ACB , ADB stand on the same arc AB , the one being at the centre and the other at the circumference.

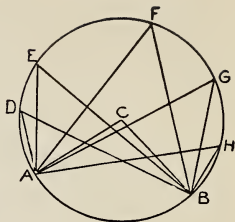


Measure the number of degrees in each, and compare these numbers.

Make the same constructions in the case of two or three other circles, and repeat the measurements and comparison.

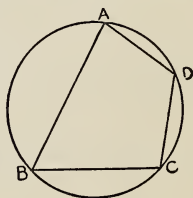
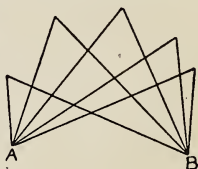
What is your conclusion as to the size of the angle at the centre, compared with the size of the angle at the circumference?

2. Construct such a figure as the annexed, where an angle ACB at the centre, and a number of angles $ADB, AEB, \dots$ at the circumference, stand on the same arc AB . Then, adjusting the bevel to the angles at the circumference, compare their magnitudes. The result of such a comparison might have been anticipated, since each of the angles at the circumference is half the same angle, ACB , at the centre.

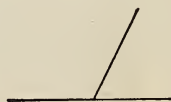


Hence angles described in the same segment of a circle, *i.e.*, angles standing on the same arc of a circle, being on the circumference, are equal to one another.

Using the bevel, construct a number of angles as in the annexed figure, all of the same magnitude, and with the sides of each passing through the points A and B . Then taking any three of the angular points, and, by the method of Ch. X., 6, constructing for the circle through these three points, show, by describing the circle, that it passes through the other angular points, and also through the points A and B .



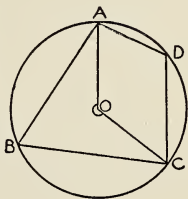
3. Take any four points, A, B, C, D , on the circumference of a circle, and join them as in the figure, so constructing a quadrilateral in the circle. Adjust the bevel to the



opposite angles **B** and **D**, and construct angles equal to them adjacent to one another. What do you observe with reference to the sum of the angles **B** and **D**? What with reference to the sum of the angles **A** and **C**?

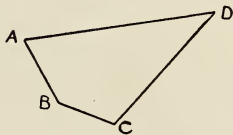
Repeat this measurement with respect to the opposite angles of other quadrilaterals in circles.

The annexed figure suggests what conclusion should be reached with respect to the sum of the angles at **B** and **D** and at **A** and **C**. For the angle marked at **O** is double of the angle **ADC** (Ch. XII., 1); and the other angle at **O** is double the angle **ABC**. Therefore the angles at **O** are together double the sum of the angles **ADC** and **ABC**. But the angles at **O** make up four right angles. Hence the angles **ABC**, **ADC** are together equal to two right angles.



Hence **the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.**

Using the protractor, construct a quadrilateral with two of its opposite angles together equal to two right angles. Taking any three of the angular points, and, by the method of Ch. X., 6, constructing for the circle through these three points, and describing the circle, note the position of the quadrilateral with respect to the circle.

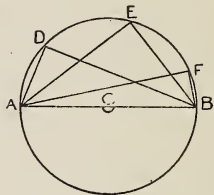


Repeat the construction for several such quadrilaterals.

The result of such observations is that if the **opposite angles of a quadrilateral are together equal to two right angles**, a circle can be described about it.

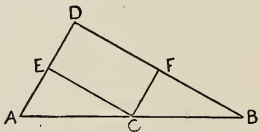
Since a quadrilateral can be divided into two triangles by joining its opposite angles, the sum of all the angles of any quadrilateral is four right angles. Hence if the sum of a pair of opposite angles be two right angles, the sum of the other pair is two right angles also.

4. Describe a circle, and in the semicircle construct a number of angles as indicated in the figure. Adjust the protractor to the angles **ADB**, **AEB**, ... What is the magnitude of these angles?

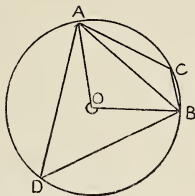


The magnitude of the angle in a semicircle may be *proved* thus: The *straight* angle **ACB** at the centre is (Ch. XII., 1) double any of the angles at the circumference. But the straight angle **ACB** is 180° . Hence **the angle in a semicircle is 90°** .

ADB being a right-angled triangle, find the centre of the circle through **A**, **D** and **B**, by bisecting **AD**, **BD** and drawing the perpendiculars **EC**, **FC**. Note that these perpendiculars intersect in **AB**; and note also that **C**, being the centre of the circle through **A**, **D** and **B**, the centre of the hypotenuse of a right-angled triangle is equidistant from the three angles of the triangle.



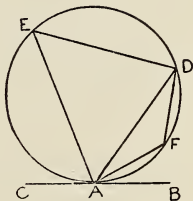
5. A chord, such as **AB**, which does not pass through the centre, divides the circle into two segments, one of which, **ADB**, is greater, and the other, **ACB**, less than a semicircle. Evidently the marked angle **AOB** is greater than two right angles, and therefore the angle **ACB**, which is half of the marked angle **AOB**, is greater than one right angle. Similarly the angle **ADB**, being half the other angle at **O**, is less than a right angle.



Hence the angle in a segment of a circle less than a semicircle is greater than a right angle; and the angle in a segment of a circle greater than a semicircle is less than a right angle.

AC, CB contain an angle which has, in succession, the magnitudes 80° , 85° , 89° , 91° , 95° . Construct in the different cases the circles through **A, C** and **B**, and note the positions of the centre with respect to the side **AB**.

6. Draw with accuracy the tangent **CAB** at any point **A** on the circumference of a circle. From **A** draw any chord **AD**, and construct the angles **AED**, **AFD** in the segments into which **AD** divides the circle. Then, using the bevel, discover the relation in size between the angle **CAD** and the angle **AFD** in the alternate segment; and the relation between the angle **BAD** and the angle **AED** in the alternate segment.



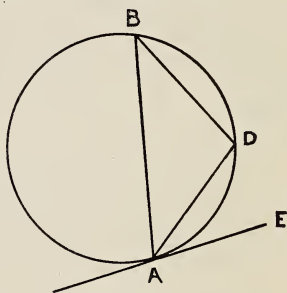
Repeat the same examination in the case of different circles, drawing the chord at various inclinations to the tangent.

As a result of these observations we are led to the conclusion that **if from the point of contact of any tangent to a circle, a chord be drawn cutting the circle, the angles the chord makes with the tangent are equal to the angles in the alternate segments of the circle.**

7. To place in a circle an angle of given magnitude, *i.e.*, to cut off from the circle a segment containing an angle of given size, we may proceed as follows:

Draw **AE** a tangent to the circle at any point **A** on its circumference. Make an angle **DAE** equal to the given angle, cutting circumference again at **D**.

Then any angle in the segment **ABD** will be equal to **DAE**, and therefore to the given angle.



Draw a circle and cut off segments containing 30° , 60° , 120° , 90° .

Summary.

1. The angle at the centre of a circle is double the angle at the circumference standing on the same arc.

2. Angles described in the same segment of a circle are equal.

3. If, on any line **AB** as base and on the same side of it any number of equal angles are drawn, a circle may be described passing through **A** and **B** and also through the vertices of all the angles.

4. The opposite angles of any quadrilateral inscribed in a circle together equal two right angles.

5. If the opposite angles of a quadrilateral be together equal to two right angles, a circle can be described about it.

6. The angle in a semicircle is a right angle. It follows that the hypotenuse is the diameter of the circle described about a right-angled triangle, and the middle point of the hypotenuse is equidistant from the three angular points.

7. The angle in a segment of a circle less than a semicircle is greater than a right angle; and the angle in a segment of a circle greater than a semicircle is less than a right angle.

8. If from the point of contact of any tangent to a circle, a chord be drawn cutting the circle, the angles the chord makes with the tangent are equal to the angles in the alternate segments of the circle.

9. To cut off from a circle a segment containing an angle of given magnitude.

Exercises.

1. Describe a circle of radius 2 inches. Draw a chord AB cutting off a segment containing an angle of 120° . What is the angle contained in the segment which forms the rest of the circle? If O is the centre of the circle, what is the magnitude of the two angles AOB? Verify by measurement.

2. In the preceding question, using the same figure, draw from B a chord BC cutting off a segment containing an angle 100° . What are the angles in the two segments cut off by CA? What are the two angles AOC?

3. Describe a circle of radius $1\frac{1}{2}$ in. Draw in it a chord AB, dividing the circle into two segments, ACB, ADB, containing angles of 70° and 110° respectively. Construct in the circle an angle CAD of 50° . What is the angle CBD? Mark on the quadrilateral ACBD the size of each angle.

4. Describe a circle of radius $1\frac{1}{2}$ in., and in it place an angle of 60° . In it also describe a triangle of vertical angle 60° and altitude 2 in.

5. Describe a circle of radius $1\frac{1}{4}$ in., and in it describe a triangle with angles 50° , 55° and 75° .

6. Describe a circle of radius $1\frac{5}{8}$ in., and in it construct a quadrilateral with angles 55° , 75° , 125° , 105° .

7. Describe a circle of radius $1\frac{1}{2}$ in., and in it draw a number of chords, AB, CD, EF, . . . all cutting off angles of 60° . Are the chords all of the same length? Apply test.

8. ABCD is a quadrilateral in a circle, and the side AB is produced to E. To what angle of the quadrilateral is the exterior angle CBE equal? Apply test. Give reasons.

9. AB is a line of length $2\frac{1}{4}$ in. If on it a segment of a circle is to be constructed containing an angle of 60° , what angle will AB subtend at the centre C? What are the angles of the triangle CAB? Find C by construction, and then describe the circle.

10. Construct on a line $2\frac{1}{2}$ in. long, a segment of a circle containing an angle of 70° . Test the accuracy of your construction by measuring an angle in the segment.

11. AB, CD are two chords, perpendicular to each other, in a circle whose centre is O. Of what angles are the angles AOC, BOD double? What, therefore, is their sum?

12. AB, CD are two chords of a circle, intersecting in E. Show that the triangles AEC, DEB are equiangular.

13. ABCD is a quadrilateral in a circle, and the sides AB, CD, produced, meet in E. Show that the triangles EBC, EDA are equiangular.

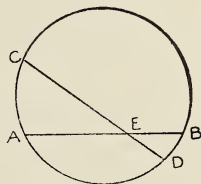
14. AB, AC are two tangents to a circle whose centre is O. Show that a circle may be described to pass through OBAC.

15. AD, BE are drawn perpendicular to the opposite sides of the triangle ABC. Show that a circle can be described about AEDB, and describe it. How are the angles ABC, DEC related? Apply test. Give reasons.

CHAPTER XIII.

Relation Between Segments of Intersecting Chords.

1. **AEB** and **CED** are any two chords in a circle, intersecting at **E**.



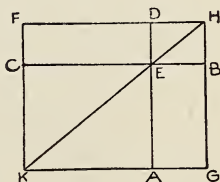
In the second figure **CEB** and **AED** are any two lines drawn perpendicular to each other, and on these we lay off the following distances with the dividers :

AE = **AE** of circle

EB = **EB** " "

CE = **CE** " "

ED = **ED** " "



Complete the rectangles **CEDF** and **AEBG**, and let **FD** and **GB**

meet in **H**. Then produce the lines **FC**, **HE** and **GA**, and note how nearly they come to passing through the same point (at **K**). Go over the measurements and construction with extreme care, getting rid of all inaccuracies. Do these lines (**FC**, **HE**, **GA**) all pass through the same point? If they do, how do the areas **CEDF**, **AEBG** compare in size (Ch. VIII., 7), and therefore the rectangles **AE.EB**, **CE.ED**, contained by the segments of the chords?

Measure each of the lines **AE**, **EB**, **CE**, **ED** in the circle, and examine whether the product of **AE** and **EB** is approximately equal to the product of **CE** and **ED**.

Describe other circles, draw two chords in each, and repeat in the case of each circle the construction of the second figure. Repeat also the measurements and multiplications.

The result of our observations may be stated as follows: **If two chords of a circle cut one another within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.**

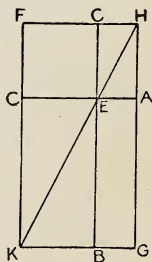
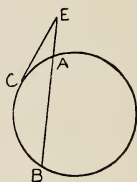
2. Draw accurately the tangent **EC**; draw also the secant **EAB**.

In the second figure **CEA**, **BEC** are any two lines drawn at right angles to each other, and on these we lay off the following distances with the dividers:

$$\begin{aligned} EA &= EA \text{ of circle} \\ EB &= EB \text{ " " } \\ EC, EC &= EC \text{ " " } \end{aligned}$$

Complete the rectangle **EBGA** and the square **ECFC**, and let **FC**, **GA** meet in **H**. Then produce the lines **FC**, **HE** and **GB**, and note how nearly they come to passing through the same point (at **K**). Go over the measurements and construction with extreme care, getting rid of all inaccuracies.

Do these lines (**FC**, **HE**, **GB**) all pass through the same point? If they do, how do the areas **EBGA**, **ECFC** compare in size (Ch. VIII., 7) and therefore the rectangle **EA.EB** and square on **EC** (see figure of circle)?

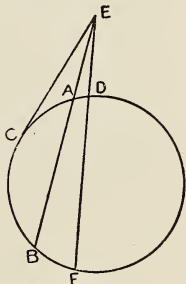


Measure each of the lines **EA**, **EB**, **EC** in the first figure, and examine whether the product of **EA** and **EB** is approximately equal to the square of **EC**.

Describe other circles, draw to each a secant and a tangent from the same point, and repeat in the case of each the construction of the second figure. Repeat also the measurements and multiplications.

The result of our observations may be stated as follows: **If from any point without a circle two straight lines be drawn, one a secant and the other a tangent, then the rectangle contained by the secant and the part of it without the circle is equal to the square on the tangent.**

If another secant **EDF** be drawn, since the rectangle contained by **EA** and **EB** is equal to the square on **EC**, and the rectangle contained by **ED** and **EF** is equal to the square on **EC**, therefore the rectangle contained by **EA** and **EB** is equal to the rectangle contained by **ED** and **EF**.



The segments of one chord are 3, 4, and of another 2, 6 quarters of an inch, the chords making an angle of 30° with one another; describe the circle through the extremities of the chords. If the segments of another line through the intersection of the chords be $1\frac{1}{2}$ and 8 quarters of an inch, do the ends of this necessarily rest on the circle? Place the line that its ends may so rest.

The tangent to a circle is $2\frac{1}{2}$ inches; a secant is $3\frac{1}{2}$ inches, and the part of it without the circle $1\frac{1}{2}$ inches. These lines make an angle of 60° with one another. Describe the circle.

Summary.

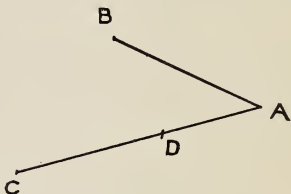
1. If two chords of a circle intersect, the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other.

2. If, from a point without a circle, a tangent and a secant be drawn, the rectangle contained by the secant and the part of it without the circle is equal to the square on the tangent.

3. If, from a point without a circle, two secants be drawn, the rectangle contained by one secant and the part of it without the circle is equal to the rectangle contained by the other secant and the part of it without the circle.

4. If two straight lines intersect so that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other, a circle may be described to pass through the extremities of these lines.

5. If two straight lines such as AB, AC be drawn from a point A, and if the longer of these, say AC, be divided at D so that the rectangle contained by AC.AD equals the square on AB, then AB will be a tangent to the circle described through B, C and D.



Exercises.

1. Two lines, AB, CD cut one another in E. $AE = 1\frac{1}{2}$, $EB = 2$, $CE = 3$, $ED = 1$ in., so that $AE \cdot EB = CE \cdot ED$. Describe a circle to pass through A, C, B, D.

2. Describe a circle of radius 2 in. Draw a diameter AB. Take in it a point C at distance 1 in. from centre, and draw chord DCE perpendicular to AB. By construction, as in text, show that rectangle AC.CB is equal to square on CD. Why should this be so? What is the length of CD? Calculate and verify by measurement.

3. Two lines intersect at an angle of 30° . The segments of one 2 in. and $\frac{1}{2}$ in., of the other, both 1 in. Describe a circle to pass through the ends of the lines. With what inclination of the lines to one another would the longer line become a diameter?

4. Describe a circle of radius 3 in. In it place a chord of length 4 in., and take in the chord a point at distance 1 in. from an end. Through this point draw another chord whose segments shall be $1\frac{1}{2}$ in. and 2 in. How many such chords may be drawn?

5. On a line take lengths AB, AC of 1 and 4 inches respectively in the same direction. Draw a line AD of 2 inches, making an angle of 45° with AC. Describe a circle through B, C, D. What is AD with respect to this circle?

6. AB, AC, measured along the same line, in the same direction, are $\frac{1}{2}$ inch and 2 inches; and AD another line through A is 1 inch. Place AD so that the circle through B, C and D may have its centre in AC.

7. Any lengths AB, AC are measured along the same line, in the same direction. Describe a number of circles passing through B and C. From A draw tangents to these circles. Measure the lengths of these tangents. Can you give any reason for what you observe?

8. AB, AC, measured along the same line, in the same direction, are 1 and 4 inches long. Describe a circle on BC as diameter and draw a line ADE cutting the circle in D and E, so that $AD = 1\frac{1}{2}$ in. What is the length of AE? Draw a tangent to the circle from A. What is its length? Verify your results by calculation.

9. Describe two circles of radii 1 and 2 inches respectively, intersecting in A and B. Draw a straight line through A and B, and from any point in it, draw a tangent to each circle. Measure the tangents. What relation between their lengths? Give reason.

10. Describe any two circles touching each other externally. Draw their common tangent at the point of contact, and from any point on it draw another tangent to each of the circles. What do you observe as to the lengths of these tangents? Why? Does this hold good if the circles touch one another internally?

11. The common chord of two intersecting circles cuts a line drawn touching both circles. Show that it bisects this line. Why?

12. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to AB. By describing a circle about ABC, reason out that the rectangle AD.DB is equal to the square on CD.

13. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to AB. By describing a circle about the triangle CDB, reason out that the rectangle AD.AB is equal to the square on AC.

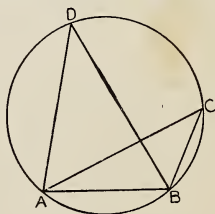
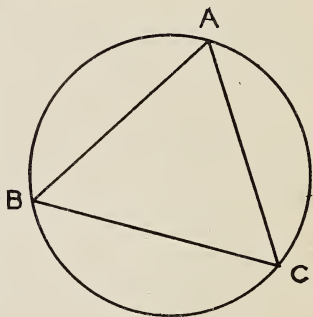
CHAPTER XIV.

Triangles In and About Circles.

1. Draw any circle, and take **ABC** any three points on its circumference. Join **AB**, **BC**, **CA**. The triangle **ABC** is said to be inscribed in the circle.

It is not always possible to inscribe in a circle a triangle of given size, but we can, as shown below, inscribe in a circle a triangle of given shape, *i.e.*, equiangular to another triangle, or a triangle with its angles of given size, their sum, of course, being 180° . Thus, let it be required to construct in a given circle a triangle with angles of given size, say, 30° , 70° , 80° .

Using the protractor, adjust the bevel to an angle equal to any one of these, say 30° . Place the angle of the bevel at any point **C** on the circumference, and with a needle mark the points, **A** and **B**, where the legs of the bevel cross the circumference. We have thus a

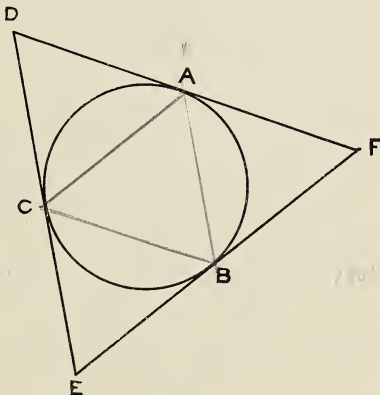


segment **ACB** containing an angle of 30° , and all angles in the segment **ACB** are angles of 30° . With the protractor at **A** make the angle **BAD** of 80° . Join **BD**. Then **ADB** is an angle of 30° . Hence the remaining angle **ABD** is 70° .

Of course the angle of 30° at **C** may be constructed with the protractor. The segment containing an angle of 30° may also be obtained by constructing at the centre an angle of 60° .

In a circle whose radius is $1\frac{1}{2}$ in., construct a triangle whose angles are 65° , 75° and 40° .

2. If any circle be drawn, and at points **A**, **B**, **C** on this circle tangents be drawn and produced to meet at points **D**, **E**, **F**, the triangle **DEF** so formed is said to be described about the circle.

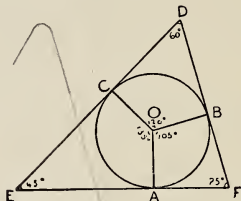


To construct a triangle whose sides shall be tangents to a given circle, and whose angles shall be of given magnitude, say, 75° , 45° and 60° .

We can scarcely here proceed as in the previous case, adjusting the legs of the bevel to, say, the angle 75° , and placing them across the circle so as to be

tangents to it. To assume that we can construct the tangent to a circle by laying the ruler against it and so drawing a line, is equivalent to assuming that we can lay off a right angle, using only the judgment of the eye.

It will be well to proceed thus: Find the angles which are the supplements of 75° , 45° and 60° , *i.e.*, 105° , 135° and 120° . Draw any radius **OA**, and make the angle **AOB** of 105° , and the angle **AOC** of 135° . The remaining angle **BOC** must be of 120° , since $105^\circ + 135^\circ + 120^\circ = 360^\circ$. Draw lines (tangents) at **A**, **B** and **C** at right angles to the radii.



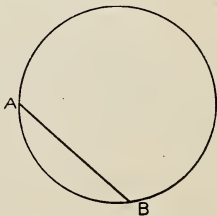
Since the angles of a quadrilateral make up four right angles, and the angles at **A** and **C** are right angles, therefore $\text{AOC} + \text{AEC} = 180^\circ$. But **AOC** is 135° .

$\therefore$ **AEC** is 45° . Similarly, angles at **D** and **F** are 60° and 75° respectively.

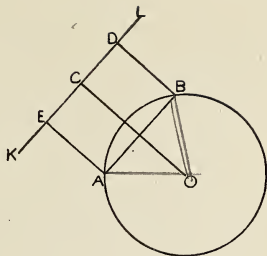
About a circle whose radius is $1\frac{1}{4}$ in., construct an equilateral triangle.

About a circle whose radius is $1\frac{1}{2}$ in., construct an isosceles triangle whose vertical angle is 30° .

3. In a circle we readily place a chord of any required length. For, take the length on the ruler with the points of the dividers, and place the points of the dividers on the circumference of the circle. The ends of the chord **A**, **B** are thus marked, and the chord can be drawn.



We can without difficulty draw the chord in a required position, for example, parallel to a given line, **KL**: Draw **OC** perpendicular to **KL**, and mark off **CD**, **CE** each equal to half the length of the chord. Then draw **DB**, **EA**, parallel to **CO**.



The chord **AB** is equal to **ED**, and therefore is of the required length, and it is parallel to **KL**.

We may draw **EA** alone perpendicular to **KL**, and then draw **AB** parallel to **KL**, thus not using the point **D** or line **DB**.

Of course the chord can never be greater than the diameter of the circle in which it is to be placed.

In a circle of radius 1 inch, place ten chords of length $\frac{1}{2}$ inch, such that each ends at the point where the next begins.

In a circle place a chord of given length so that it may be perpendicular to a given line.

Summary.

1. To inscribe a triangle in a circle, equiangular to a given triangle.
2. To describe a triangle about a circle, equiangular to a given triangle.
3. In a given circle to place a chord of given length and given direction.

Exercises.

1. In a circle of radius $1\frac{1}{2}$ in., construct an equilateral triangle.
2. In a circle of radius 2 in., construct an isosceles triangle, the angle at the vertex being 55° .

3. In a circle of diameter $3\frac{1}{2}$ in., construct an equilateral triangle, such that its base shall be parallel to the top or bottom of your paper. (Draw a line through centre perpendicular to top or bottom of paper, and at centre construct, on each side of this line, angles of 60° , etc.)

4. Construct a triangle with angles of 55° , 65° , and 60° , and in a circle whose radius is $1\frac{3}{4}$ in. construct a triangle equiangular to this, its sides being also parallel to the sides of this triangle.

5. Describe a circle of radius $1\frac{1}{2}$ inches, and draw a line making an angle of 60° with the side of your paper. Draw a tangent to the circle parallel to this line. (From centre drop a perpendicular on the line. This gives point through which tangent is to be drawn.)

6. Describe a circle of radius $1\frac{1}{4}$ inches. Draw a line making an angle of 75° with the top or bottom of your paper, and draw a tangent to the circle perpendicular to this line.

7. Describe a circle of radius $1\frac{1}{4}$ inches, and about it describe an equilateral triangle so that two of the sides may make angles of 60° with the side of your paper, the third side being parallel.

8. Describe a circle of radius 1 inch, and about it describe an isosceles triangle whose vertical angle is 40° , the base of the triangle being parallel to the top or bottom of your paper.

9. About a circle of radius $1\frac{1}{4}$ in. describe a triangle whose angles are 30° , 70° and 80° .

10. Three lines intersect to form a triangle. Describe a circle of radius $1\frac{3}{16}$ in., and about it describe a triangle whose sides are parallel to the lines. Test the accuracy of your construction by comparing the angles of the two triangles.

11. Describe a circle of radius $1\frac{1}{2}$ in. In and about it describe two triangles with angles 50° , 60° and 70° , so that corresponding sides are parallel to each other.

12. An equilateral triangle is inscribed in a circle, and another is described about the circle. What relation exists between the lengths of the sides?

13. Describe a circle of radius $1\frac{3}{8}$ inches, and draw two tangents to it, such that the angle between them is 25° .

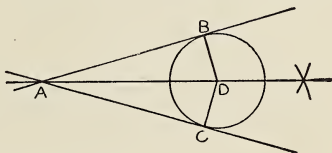
14. In a circle of radius 1 in., place in succession four chords, AB, BC, . . . , each of length $\sqrt{2}$ in. (See Chapter IX.)

15. Describe two circles of radii 1 in. and 2 in. In them describe triangles with angles of 45° , 65° and 70° . Compare the lengths of corresponding sides of the two triangles. What should the sides be if the radius is $\frac{1}{2}$ inch? 3 inches? Verify by constructing the figures.

CHAPTER XV.

Circles In and About Triangles.

1. If the angle **BAC**, between two lines, be bisected, and, from any point **D** in it, perpendiculars **DB**, **DC** be drawn, these perpendiculars are evidently equal. If, then, a circle be described with centre



D, and radius **DB** or **DC**, it will touch both the lines.

Thus all circles touching both lines have their centres in the straight line which bisects the angle between the lines.

Two lines make an angle of 120° with one another. Describe four circles, of different radii, touching both of them.

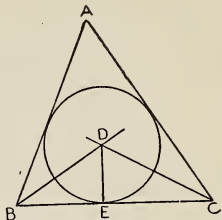
Two lines make an angle of 80° with one another. Describe a circle of radius $\frac{1}{2}$ in. to touch both of them; also of radius 1 in.

Two lines make an angle of 60° with one another. Describe a circle touching both of them; also a second circle touching the previous circle and the two lines.

2. **We may describe a circle touching the three sides of a triangle as follows:**

Bisect the angles at **B** and **C** by the lines **BD**, **CD**.

Then **BD** contains the centres of circles touching **BA** and **BC**; and **CD** contains the centres of circles touching **CA** and **CB**. Hence **D** is the centre of a circle which touches all three sides. **DE**, perpendicular to **BC**, is the radius of this circle. Hence with centre **D** and radius **DE**, describe a circle. This is the circle *inscribed* in the triangle **ABC**.



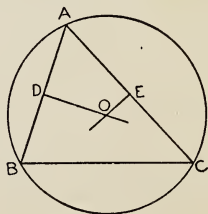
The utmost care is to be exercised in accurately bisecting the angles; otherwise it may be found that, when the circle is described, it cuts a side, or falls short of one.

Inscribe a circle in the triangle whose sides are 3, $3\frac{1}{4}$ and $3\frac{1}{2}$ inches.

Describe a circle to touch the other side of **BC** (any triangle **ABC**), and the sides **AB** and **AC** produced.

The base of a triangle is 2 in., and the angles at the base are 40° and 110° . Inscribe a circle in it. Measure its radius.

3. We have already (Ch. X., 6), in effect, shown **how to describe a circle about any triangle**, *i.e.*, to pass through the angular points of the triangle. Two sides, say **AB** and **AC**, are bisected, and **DO**, **EO** are drawn through the points of bisection perpendicular to **AB** and **AC**, respectively. Then all points in **DO** are equally distant from **A** and **B**; and all points in **EO** are equally distant from **A** and **C**. Hence



O is equally distant from **A**, **B** and **C**; and if the sharp point of the compasses be placed at **O**, and the pencil end at **A**, or **B**, or **C**, and a circle be described, it will pass through **A**, **B** and **C**.

Here again the greatest care must be exercised in bisecting the sides, and in drawing the perpendiculars at the points of bisection; otherwise the circle will pass through the angle on which the pencil end of the compasses was placed, but may not pass through the two other angles.

Describe a circle about a triangle whose sides are 2, $2\frac{1}{2}$ and 3 in. Measure its radius.

The side of an equilateral triangle is 3 in.; describe a circle about it.

Each of the equal sides of an isosceles triangle is 3 in., and the equal angles are each 75° . Describe a circle about it.

Summary.

1. Any point on the bisector of an angle is equidistant from the arms of the angle, and with any point on this bisector as centre a circle may be described touching the arms.

2. To inscribe a circle in a given triangle.

3. To describe a circle touching one side of a triangle and the other sides produced. (This circle is called an *escribed* circle to the triangle.) How many such circles may a triangle have?

4. The perpendicular bisector of a line is equidistant from the ends of the line; hence, with any point on the perpendicular bisector as centre, a circle may be described passing through the extremities of the line.

5. To describe a circle about a triangle.

Should the course contained in this book prove too long for a year's work, it is suggested that Chapters XVI. and XVII. be omitted, valuable though they may be as affording exercises in accurate geometrical construction.

Exercises.

1. Draw two lines making an angle of 50° with one another, and describe three circles touching both lines.

2. Describe a triangle with sides 3 in., $2\frac{3}{4}$ in., $1\frac{3}{4}$ in., and in it inscribe a circle.

3. In the case of the triangle of the previous question, describe circles touching each side and the other two sides produced.

4. Having obtained the four circles of the two previous questions, through what points do the lines joining any two centres pass? What position does the line joining any two centres occupy with respect to the line joining the other two centres? Apply tests in both cases.

5. Two parallel lines are $1\frac{1}{2}$ in. apart, and a third line cuts them at an angle of 60° . Describe all the circles you can, each touching the three lines. What is the length of the radius?

6. In the previous question, what is the figure formed by joining the centres to the points where the parallels are cut by the third line? Apply test.

7. Is there any position which three lines can occupy, such that no circle can be described touching all?

8. Describe an equilateral triangle with side 2 in., and in it inscribe a circle. Express with exactness the radius of this circle.

9. Describe also a circle about the triangle of the previous question, and express with exactness its radius.

10. Draw a triangle ABC with AB $1\frac{1}{2}$ ins., BC 2 ins. and AC $2\frac{1}{2}$ ins. long. What kind of a triangle is it? Describe a circle about it. Where does the centre come and why?

11. Draw a triangle with angles 105° , 30° , 45° , and describe a circle about it. Note the position of the centre. Can you give any reason for this?

12. Is there any position which three points can occupy with respect to one another, such that a circle cannot be described to pass through all?

13. Two lines make an angle of 70° with one another. Describe a circle of radius $1\frac{1}{2}$ in. touching both of them. (Draw a perpendicular to either of the lines, of length $1\frac{1}{2}$ in., and through its end draw a line parallel to the line on which the perpendicular stands, producing this parallel until it meets the bisecting line.)

14. ABCD is a quadrilateral; $A=85^\circ$, $B=80^\circ$, $C=95^\circ$; $AB=2\frac{1}{2}$ inches, $BC=3\frac{1}{4}$ inches. Construct the quadrilateral and describe a circle about it. Is it possible to describe a circle about any quadrilateral?

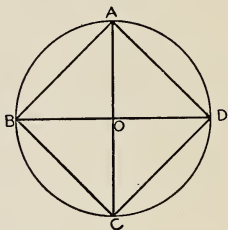
15. Draw a circle and at four points on its circumference draw tangents. Let these tangents be produced to meet, forming a quadrilateral. Compare the sum of one pair of opposite sides with the sum of the other pair. Why is this?

16. Draw a quadrilateral, such that a circle may be inscribed in it. Inscribe the circle in this quadrilateral.

CHAPTER XVI.

Squares and Circles In and About Circles and Squares.

1. To inscribe a square in a circle, draw two diameters at right angles to one another and join their extremities. The construction being accurately made, the set-square will show that the angles **A**, **B**, **C**, **D** are all right angles; and the equality of the sides **AB**, **BC**, . . . may be proved by using the dividers.



Of course the evident equality of the triangles **AOB**, **BOC**, . . . proves the equality of the sides, and the angles **ABC**, **BCD** . . . are all right angles, because they are angles in semicircles.

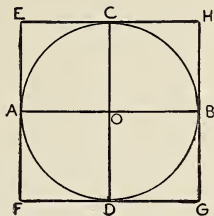
Inscribe a square in a circle of radius $1\frac{1}{2}$ inches. Test the accuracy of your construction by examining, with the dividers, the equality of the sides.

In the figure above, if **AC** and **BD** are not at right angles to each other, what sort of figure is **ABCD**?

In a circle whose radius is 3 inches, inscribe a rectangle, one of whose sides is 1 inch. Test with instruments.

Note.—If a square is inscribed in a circle, the diameter of the circle is the diagonal of the square.

2. To describe a square about a given circle, draw two diameters at right angles to each other, and through the ends of each diameter draw lines parallel to the other. The construction being accurately made, the set-square will show that the angles of **E, F, G, H** are all right angles, and the equality of the sides **EF, FG, . .** may be proved by using the dividers.



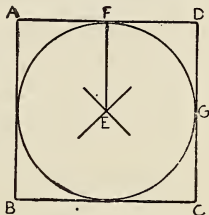
Evidently the figures **AOCE, AODF, ...** are equal squares, whence we readily prove that the sides of **EFGH** are all equal; and its angles are right angles.

Describe a square about a circle whose radius is $1\frac{1}{2}$ inches. Test the accuracy of your construction by finding whether the sides are equal, using the dividers; and use the set-square to determine whether the angles are right angles.

Draw two diameters in a circle not at right angles to each other, and draw tangents at their extremities. Determine the nature of the figure formed by the tangents by measuring the lengths of its sides.

Draw a circle of radius 1 inch, and about it describe a rhombus containing an angle 60° .

Note.—If a square is described about a circle, the diameter of the circle is equal to the side of the square. Does this hold true regarding the rhombus?



3. To inscribe a circle in a given square, draw portions of the diagonals of the square, so that they intersect, as at **E**. Draw **EF** perpendicular to

one of the sides. With **EF** as radius, describe a circle. If the construction has been accurate, the circle will touch the sides of the square.

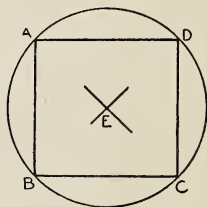
By drawing the complete diagonals it may readily be shown, from the equality of such triangles as **EFG**, **EGD**, that the perpendiculars from **E** on the sides are equal.

Describe a square with side of 4 inches, and in it inscribe a circle. Show, by measurement with dividers and set-square, that the lines joining the points of contact form a square. Show that the sides of this are perpendicular to the diagonals of the original square.

Inscribe a circle in the second square **FEGD** of the preceding question.

Inscribe a circle in a rhombus, each of whose sides is 4 inches, and one of whose angles is 60° .

4. To describe a circle about a given square, draw portions of the diagonals so that they intersect. Then, placing the sharp point of the compasses at **E**, where the diagonals intersect, and the pencil point on any one of the angles, and describing a circle, it will pass through the other angular points of the square.



The lines from **E** to the angles are equal if the square has been accurately constructed and the diagonals accurately drawn ; for the diagonals of all parallelograms bisect each other, and the diagonals of a square are equal.

Construct squares of sides $1\frac{1}{2}$, $2\frac{1}{4}$ and 3 inches, and about each describe a circle.

Draw a rectangle whose sides are 1 and 3 inches, and about it describe a circle.

Summary.

1. To inscribe a square in a given circle.
2. To inscribe a rectangle in a given circle.
3. To describe a square about a given circle.
4. To describe a rhombus about a given circle.
5. To inscribe a circle in a given square.
6. To inscribe a circle in a given rhombus.
7. To describe a circle about a given square.
8. To describe a circle about a given rectangle.
9. If a square or rectangle be inscribed in a circle, the diameter of the circle is equal to the diagonal of the square or rectangle.
10. If a square or rhombus be described about a circle, the diameter of the circle is equal to the side of the square or rhombus.

Exercises.

1. Draw a circle of radius 1 inch. What is the length of the side of the square (*a*) described about it, (*b*) inscribed in it?
2. What is the area of each of the squares in 1?
3. What would be the radius of a circle inscribed in a square 3 ins. to the side? What would be the radius of the circle described about this square?
4. Is it possible to describe a rectangle (not a square) about a circle?
5. Is it possible to inscribe a rhombus in a circle? Can you give any reason?
6. About a circle of radius $1\frac{1}{4}$ ins. describe a quadrilateral with angles 60° , 150° , 110° , 40° .
Can you describe about a circle a quadrilateral equiangular to any given quadrilateral?
7. In a circle of radius $1\frac{1}{4}$ in. describe a quadrilateral with angles 60° , 150° , 120° , 30° .

Can you inscribe in a circle a quadrilateral equiangular to any given quadrilateral?

CHAPTER XVII.

Regular Polygons.

1. A **polygon** is a rectilineal figure contained by more than four straight sides.

A **pentagon** is a figure of 5 sides.

hexagon " " 6 "

heptagon " " 7 "

octagon " " 8 "

decagon " " 10 "

dodecagon " " 12 "

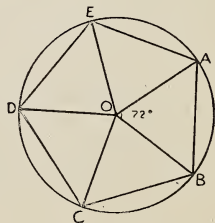
quindecagon " " 15 "

A polygon is said to be **regular** when all its sides are equal, and also its angles equal.

2. The angles at any point, for example, at the centre of a circle, make up 360° . We can divide this interval, by means of the protractor, into a number, 5, 6, 8, ..., of equal angles. If we prolong the sides of these angles until they intersect the circumference of the circle, and join the successive points of intersection, we have a **regular polygon** of 5, 6, 8, ... sides, as the case may be.

3. To describe a regular polygon in a circle :

A pentagon having five sides, the angle subtended at the centre of the circle by the side of a regular pentagon



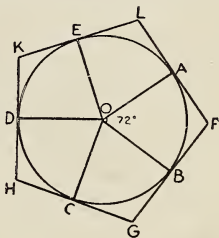
inscribed in the circle, will be $\frac{1}{5}$ of $360^\circ = 72^\circ$. Using then the protractor, or adjusting the bevel to an angle of 72° , lay off at the centre 5 angles, each of this magnitude. Produce the sides of the angles to meet the circumference, and join the succeeding points of intersection. The construction being accurately made, the bevel will show the equality of the angles **ABC**, **BCD**, . . . , and the dividers will show the equality of the sides **AB**, **BC**.

Of course, the evident equality of the isosceles triangles **OAB**, **OBC**, . . . , proves the equality of the sides and angles of the pentagon.

The angle at the vertex of each isosceles triangle in the figure being 72° , each angle at the base must be 54° ; and therefore each of the angles (**ABC**, **BCD**, . . .) of a regular pentagon is 108° .

4. If tangents to the circle be drawn at the angular points of the pentagon **ABCDE**, the tangents form another regular pentagon, which is said to be about the circle.

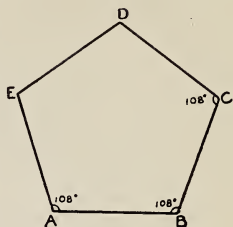
The equality of the sides **FG**, **GH**, . . . may be tested with the dividers, and the equality of the angles **FGH**, **GHK**, . . . with the bevel.



5. If we wish to construct on a given straight line (**AB**), as side, a regular pentagon, at the points **A** and **B**, with the protractor we mark off angles **BAE**, **ABC** of 108° , and with the dividers make **BC** and **AE**, each equal to **AB**. At **C** we again make an angle **BCD** of 108° , and mark off **CD** equal to **AB**. Joining **E** and **D**,

we have a regular pentagon **ABCDE**. Using the bevel, we shall find that the angles at **E** and **D** are equal to the three other angles, and the dividers will prove the side **DE** to be equal to the other sides.

The radius of a circle being $1\frac{1}{2}$ inches, inscribe in it a regular pentagon. With the dividers and bevel prove the accuracy of your construction, — that the sides and angles are equal.

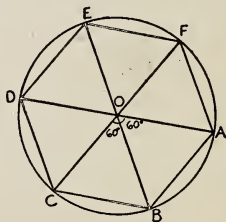


Describe also about the same circle a regular pentagon. With the dividers and bevel prove the accuracy of your construction.

On a line of length 2 inches, as side, construct a regular pentagon. With instruments prove the accuracy of your construction.

6. To inscribe a regular hexagon in a circle:

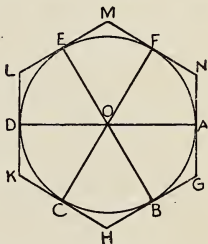
A hexagon having 6 sides, the angle subtended at the centre of the circle by the side of a regular hexagon inscribed in the circle, will be $\frac{1}{6}$ of $360^\circ = 60^\circ$. Using then the protractor, or adjusting the bevel to an angle of 60° , lay off at the centre two angles of 60° . Produce the three sides of these angles both ways to the circumference, and join the succeeding points of intersection. The construction being accurately made, the bevel will show the equality of the angles **ABC**, **BCD**, . . . , and the dividers will show the equality of the sides **AB**, **BC**, ...



Since, however, each of the triangles in the figure is equilateral, having its sides equal to the radius, the sides of the hexagon are equal to the radius of the circle. **Hence the easiest way to describe a hexagon in a circle is to measure off, with the dividers, six chords in succession, each equal to the radius.**

Evidently the angle of a regular hexagon is 120° .

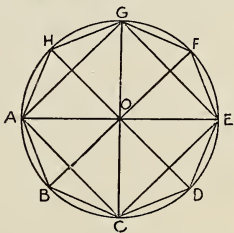
7. If tangents to the circle be drawn at the angular points of the hexagon $ABCDEF$, the tangents form another hexagon, which is said to be **about the circle**. The equality of the sides $GH, HK, \dots$ may be tested with the dividers, and the equality of the angles $GHK, HKL, \dots$ with the bevel.



8. If we wish to construct a regular hexagon with sides of given length, we describe a circle with radius of this length, and in it inscribe a regular hexagon as in § 6.

9. **To inscribe a regular octagon in a circle:**

We may construct at the centre eight angles, each of 45° , and join the ends of consecutive radii bounding these angles; or, perhaps more conveniently, we may proceed as follows: Draw two diameters at right angles to one another and join their extremities. We



thus have a square in the circle. Through the centre, using parallel rulers, draw diameters parallel to the sides

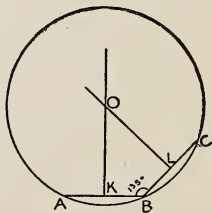
of the square. The quadrants are thus bisected, and we get eight equal angles at the centre. Joining ends of the successive radii which bound these angles, we have an octagon inscribed in the circle. The accuracy of the construction may be tested by using the dividers to determine whether the sides are equal, and the bevel to determine whether the angles are equal.

Each of the angles at the centre is 45° . Hence each of the angles at the base of any of the isosceles triangles, OAB , OBC , . . . is $67\frac{1}{2}^\circ$, and the angle of a regular octagon is 135° .

10. If tangents be drawn at the angular points of the octagon $ABCDEFGH$, the tangents form another regular octagon which is said to be about the circle.

11. To describe a regular octagon with side, AB , of given length we may proceed as follows:

Construct the angle ABC of 135° , and make $BC = AB$. Bisect AB and BC in K and L , and draw KO , LO perpendicular to AB and BC . With O as centre, and radius OA , OB or OC describe a circle. On this lay off with the dividers six chords equal to AB or BC , beginning at the point C or A . That the rest of the circle is exactly taken up with six such chords affords a test of the accuracy with which the angle ABC (135°) is constructed, AB and BC are bisected, and the perpendiculars KO and LO are drawn.



12. The pupil may continue these exercises, constructing regular decagons, dodecagons, etc., in a way quite analogous to the preceding constructions.

Summary.

1. To inscribe any regular polygon within a circle, divide 360° by the number of sides in the polygon, and at the centre of the circle make angles equal to the quotient. The points where the arms of these angles cut the circumference will be the angular points of the figure. Join these points in succession.

2. To describe a regular polygon about a circle, proceed as in (1), but instead of joining the points where the arms of the angles cut the circumference, at these points draw tangents to the circle. Produce these tangents to meet and the figure required will be formed.

3. To describe a regular polygon on a side AB of given length. Determine the magnitude of each of the interior angles of the required figure (see Chapter V., 5). Make angle ABC equal to this angle, making $BC=AB$. Through A, B, C describe a circle. Lay off on the circumference distances CD, DE, etc., each equal to AB. ABCDE, etc., will be the required figure.

4. If a regular hexagon is inscribed in a circle, the radius of the circle equals the side of the hexagon. Hence, to inscribe a regular hexagon in a circle, lay off on the circumference successive chords equal in length to the radius.

Exercises.

1. In a circle of radius $1\frac{1}{4}$ in., inscribe a regular pentagon.
2. About a circle of radius $1\frac{1}{2}$ in., describe a regular pentagon.
3. Describe a regular pentagon on a side of 1 inch.
4. In a circle of radius $1\frac{1}{2}$ in., inscribe a regular hexagon.
5. Describe a regular hexagon on a side of 1 inch.
6. Inscribe a regular octagon in a circle of radius $1\frac{1}{4}$ inches.
7. The radius of a circle is 2 inches. What is the side of a regular hexagon (a) inscribed in it, (b) described about it?
8. A regular hexagon has a side of 2 in. What is the radius of the circle (a) described about it, (b) inscribed in it?
9. On a side $\frac{1}{2}$ in. long describe a regular quindecagon.
10. Draw a regular pentagon ABCDE. Join AC, AD, BE, BD, CE.

(a) What sort of triangles are EAD, EAB, ABC, CBD, DEC? Why?

(b) What size are the angles at the bases of these triangles?

(c) What are the angles at the base of the triangle ADC? Compare them with the angle DAC at the vertex.

(d) What sort of triangles are ADC, BED, CAE, DAB, EBC?

11. In the preceding question erase the lines AB, BC, CD, DE, EA. The resulting figure (a five-pointed star) is called a pentagram.

12. Draw accurately a pentagram.

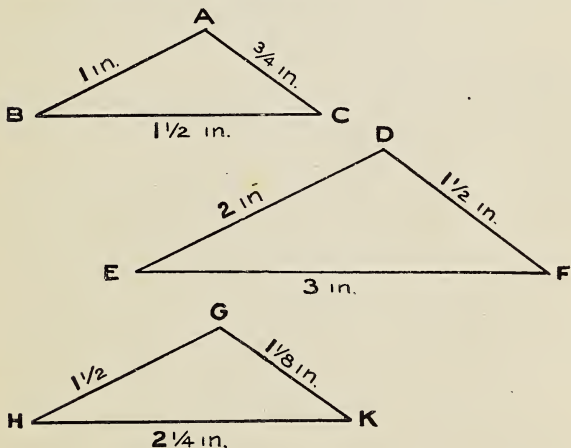
CHAPTER XVIII.

Similar Triangles.

1. Draw several triangles whose angles are 45° , 65° , 70° . Are they all of the same size? Are they all of the same shape?

Try the same experiment with triangles whose angles are 30° , 70° , 80° .

Triangles which have three angles of one respectively equal to three angles of the other are similar triangles.



2. On a base **BC** of length $1\frac{1}{2}$ inches describe a triangle **ABC**, making **AB** 1 inch and **AC** $\frac{3}{4}$ inch long.

Draw **EF** 3 inches long. Make the angle **DEF** = the angle **ABC**, and angle **EFD** = angle **ACB**. The triangle **DEF** so formed will, of course, be equiangular and therefore similar to the triangle **ABC**. Draw **HK** $2\frac{1}{4}$ in. long. Make the angle **GHK** = the angle **ABC**, and **GKH** = the angle **ACB**. The triangle **GHK** will therefore be equiangular and similar to the triangle **ABC**.

Measure very carefully the sides **DE**, **DF**, **GH**, **GK**. If measurements are accurate we should have results as follows:—

$$\begin{array}{lll} \text{AB} = 1 \text{ in.} & \text{DE} = 2 \text{ in.} & \text{GH} = 1\frac{1}{2} \text{ in.} \\ \text{BC} = 1\frac{1}{2} \text{ in.} & \text{EF} = 3 \text{ in.} & \text{HK} = 2\frac{1}{4} \text{ in.} \\ \text{AC} = \frac{3}{4} \text{ in.} & \text{DF} = 1\frac{1}{2} \text{ in.} & \text{GK} = 1\frac{1}{8} \text{ in.} \end{array}$$

$$\begin{array}{l} \text{Now} \quad \frac{\text{AB}}{\text{DE}} = \frac{1}{2}. \quad \frac{\text{BC}}{\text{EF}} = \frac{1\frac{1}{2}}{3} = \frac{1}{2}. \\ \quad \frac{\text{AC}}{\text{DF}} = \frac{\frac{3}{4}}{1\frac{1}{2}} = \frac{1}{2}. \end{array}$$

That is, in the triangles **ABC** and **DEF**,

$$\frac{\text{AB}}{\text{DE}} = \frac{\text{BC}}{\text{EF}} = \frac{\text{AC}}{\text{DF}}.$$

$$\begin{array}{l} \text{Again,} \quad \frac{\text{AB}}{\text{GH}} = \frac{1}{1\frac{1}{2}} = \frac{2}{3} \\ \quad \frac{\text{BC}}{\text{HK}} = \frac{1\frac{1}{2}}{2\frac{1}{4}} = \frac{6}{9} = \frac{2}{3} \\ \quad \frac{\text{AC}}{\text{GK}} = \frac{\frac{3}{4}}{1\frac{1}{8}} = \frac{6}{9} = \frac{2}{3}. \end{array}$$

That is, in the triangles **ABC** and **GHK**,

$$\frac{\text{AB}}{\text{GH}} = \frac{\text{BC}}{\text{HK}} = \frac{\text{AC}}{\text{GK}}.$$

In the same way it may be shown that in the triangles **DEF** and **GHK**,

$$\frac{\text{DE}}{\text{GH}} = \frac{\text{EF}}{\text{HK}} = \frac{\text{DF}}{\text{GK}}.$$

Sides which are opposite equal angles in the different triangles are called **corresponding sides**, *e.g.*, **AB, DE, GH**, are corresponding sides, as also are **BC, EF, HK**, and **AC, DF, GK**.

We may summarize the results obtained above as follows :—

If two triangles are similar, and a, b, c represent the three sides of one, and a', b', c' the corresponding sides respectively of the other, then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

The three sides of one triangle are said to be proportional respectively to the corresponding sides of the other triangle.

In the second case above $\frac{2}{3}$ or 2:3 is said to be the **ratio of corresponding sides**.

3. Construct two triangles **ABC** and **DEF**, making **AB** 1 inch, **BC** $1\frac{1}{2}$ inch, **AC** $1\frac{3}{4}$ inch, **DE** $1\frac{1}{2}$ inch, **EF** $2\frac{1}{4}$ inches, **DF** $2\frac{5}{8}$ inches.

It will be noticed that the lengths have been so chosen that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, *i.e.*, the three sides of one triangle are proportional respectively to the corresponding sides of the other triangle.

Measure carefully the angles of each triangle and compare corresponding angles. What can be said with regard to these two triangles? Test in the same way two other triangles so constructed that the three sides of one are proportional respectively to the three sides of the other.

4. Ex. 1. Two triangles are similar, the three sides of one being 1, 2 and 3 inches, and the side corresponding to the 1-inch side in the second triangle is 5 inches. Find the remaining sides of the second triangle.

Calling the triangles **ABC** and **DEF**.

Given **AB** = 1 inch,
BC = 2 inches,
AC = 3 inches,
 also **DE** = 5 inches.

Since the triangles are similar,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

$$\therefore \frac{BC}{EF} = \frac{AB}{DE} \quad \frac{2}{EF} = \frac{1}{5}.$$

$$\therefore EF = 10 \text{ inches.}$$

$$\text{Also, } \frac{AC}{DF} = \frac{AB}{DE} \quad \frac{3}{DF} = \frac{1}{5} \quad DF = 15 \text{ inches.}$$

Ex. 2. **ABC** is a triangle, and **AB** is $1\frac{1}{2}$ inches long, **BC** 2 inches, **AC** 3 inches. At a point **D** in **AB**, $\frac{1}{2}$ inch from **A**, a straight line **DE** is drawn parallel to **BC**, meeting **AC** in **E**. Find the lengths of **DE** and **AE**.

It can be seen that triangles **ADE** and **ABC** are equiangular and therefore similar.

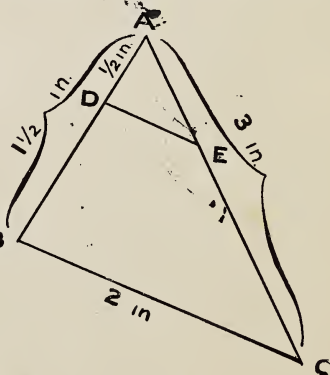
$$\therefore \frac{AE}{AC} = \frac{AD}{AB} \quad \therefore \frac{AE}{3} =$$

$$\frac{\frac{1}{2}}{1\frac{1}{2}} = \frac{1}{3}$$

$$\therefore AE = \frac{1}{3} \times 3 = 1 \text{ inch}$$

$$\text{Again, } \frac{DE}{BC} = \frac{AD}{AB} \quad \therefore \frac{DE}{2} = \frac{\frac{1}{2}}{1\frac{1}{2}} = \frac{1}{3}$$

$$\therefore DE = \frac{1}{3} \times 2 = \frac{2}{3} \text{ in.}$$



Summary.

1. Two triangles are similar when the three angles of one triangle are respectively equal to the three angles of the other.

2. If two triangles are similar, the three sides of one are proportional respectively to the corresponding sides of the other.

3. If two triangles have three sides of the one proportional respectively to the three sides of the other, then these two triangles are equiangular and therefore similar.

Exercises.

1. The sides of two triangles are $\frac{1}{2}$, $\frac{3}{4}$ and $1\frac{1}{2}$ inches, and 1, $1\frac{1}{2}$ and 3 inches respectively. Construct them and show that they are equiangular.

2. The bases of two triangles are 35 and 60 sixteenths of an inch, and the angles adjacent to the base in each triangle are 75° and 70° . Construct the triangles and show that the ratio of corresponding sides is 35:60.

3. Two sides of one triangle are $\frac{3}{4}$ inch and 1 inch, two sides of another triangle are $1\frac{1}{2}$ inches and 2 inches. The included angle in each triangle is 90° . Construct the triangles and show by measurement that they are similar. What is the ratio of the remaining sides?

4. The angles at the vertices of two triangles are both 36° . The sides adjacent to the vertex of one triangle are $1\frac{3}{4}$ in. and 2 in., and adjacent to the vertex of the other $2\frac{3}{4}$ in. and 3 in. Construct the triangles. Show by measurement that angles opposite corresponding sides are equal, and that the remaining sides are in ratio $1:1\frac{1}{2}$.

5. Two similar triangles are drawn. Given that the sides of one are 2, 3, 4 inches, and the side corresponding to the 2-inch side in the second triangle is 5 inches, calculate the remaining sides of this triangle.

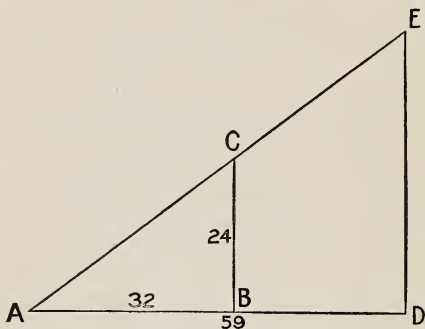
6. ABC is a triangle whose sides are $AB = \frac{3}{4}$ in., $BC = 2$ in., $AC = 1\frac{1}{2}$ in. In AB a point D is taken $\frac{1}{4}$ in. from B, and through D a line is drawn parallel to BC, cutting AC in E. Calculate the lengths of AE and DE.

7. In the preceding question, if AB is produced and the point D is taken $\frac{1}{4}$ in. below B on AB produced, and if DE is drawn parallel to BC to meet AC produced in E, calculate the lengths of AE and DE.

CHAPTER XIX.

Similar Triangles. (Continued.)

1. In the annexed figure the triangles **ABC**, **ADE** are similar. Suppose the values of the lines are



AD = 59, **AB** = 32, **BC** = 24, and that **DE** is unknown. The property of similar triangles gives

$$\frac{DE}{CB} = \frac{DA}{AB}$$

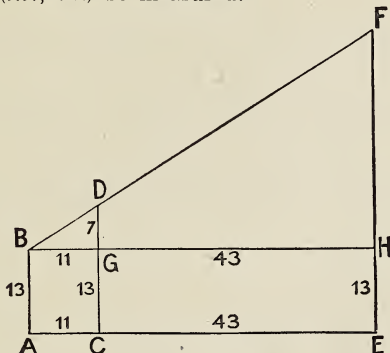
$$\frac{DE}{24} = \frac{59}{32}$$

$$DE = \frac{59}{32} \times 24 = 44\frac{1}{4}$$

2. If level ground can be found extending out from the base of a tree, or other vertical object, its height may be found as follows :

Let two rods, **AB** and **CD**, be placed upright in the ground, at such distance apart that the eye sees the tops (**B** and **D**) of the rods and the top (**F**) of the tree in the same straight line.

The heights of the rods being measured, their difference **DG** is known. Let also the lengths **AC** (*i.e.*, **BG**) and **CE** (*i.e.*, **GH**) be measured.



Suppose $AC = BG = 11$, $CE = GH = 43$,
 $AB = 13$, $CD = 20$.

Then by similar triangles **BGD**, **BHF**

$$\frac{HF}{DG} = \frac{BH}{BG}$$

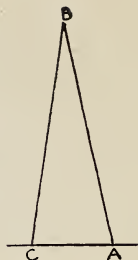
$$\frac{HF}{7} = \frac{54}{11}$$

$$HF = \frac{54 \times 7}{11} = 34\frac{4}{11}$$

Then height of object $= 34\frac{4}{11} + 13 = 47\frac{4}{11}$.

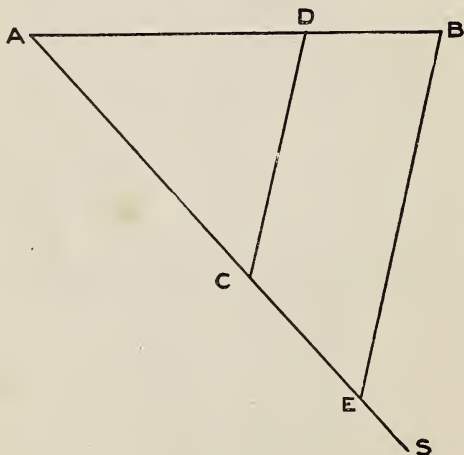
3. Suppose we wish to find the distance of an object **B** from **A**, without going over the distance **AB** with a surveyor's chain or other instrument for measuring.

Measure a *base line*, **AC**, of, say, 250 feet, and note the angles **CAB**, **ACB**. Then, on paper, construct a triangle **A₁B₁C₁**, equiangular to **ABC**, but with a base line **A₁C₁** of, say, 1 foot. Measure the length, in feet, of **A₁B₁**. The line **AB** will be 250 times the length of **A₁B₁**.



This example embodies the principle of the range-finder, so much used in military and naval operations.

4. To divide a given line in a given ratio, draw any line **AB**, say, $2\frac{1}{2}$ inches long.



Suppose we wish to divide **AB** in ratio 2:1.

Draw any line **AS** making an angle with **AB**. On **AS** mark off **AC** 2 inches, and **CE** 1 inch long. Join **BE**. Draw **CD** parallel to **BE**, meeting **AB** at **D**.

$$\text{Then } \frac{AD}{DB} = \frac{2}{1}.$$

For triangles ADC and ABE are similar.

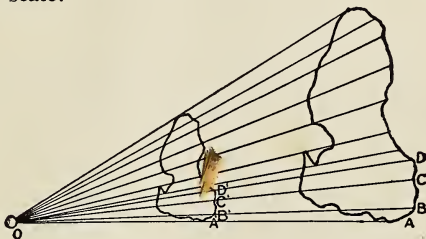
$$\therefore \frac{AD}{AB} = \frac{AC}{AE}.$$

$$\therefore \frac{AD}{AB} = \frac{2}{3}.$$

That is, of such parts as AB contains 3, AD will contain 2, and DB will contain 1.

$$\therefore \frac{AD}{DB} = \frac{2}{1}.$$

5. The proportionality of the sides of similar triangles may be employed to reduce or enlarge a figure to any scale.



Suppose we wish to obtain a figure the same shape as $ABC \dots$, but with linear dimensions half those of $ABC \dots$. Take a line $OA'A$, with $OA' = A'A$. From O draw a number of lines $OA, OB, \dots$. With the parallel rulers obtain B' , through $A'B'$ being parallel to AB ; also C' , through $A'C'$ being parallel to AC ; also D' , through $A'D'$ being parallel to AD ; and so on. Then, with the judgment of the eye, fill in the contour between A' and B' similarly to that between A and B ; between B' and C' similarly to that between

B and **C**; and so on. Any two points in the larger figure should be just twice as far apart as the two corresponding points in the smaller, and this may be used to test the accuracy of the drawing.

Maps may, in this way, be reduced or enlarged, the first drawing being obtained by using translucent paper, or by tracing against a window pane. Of course the drawing of all maps is, in part, a question of the construction of similar figures.

Exercises.

1. Draw any line AB and divide it in the ratio 7:8.
2. A line is $4\frac{1}{2}$ in. in length. Divide it into three parts, such that they shall be to one another as 7:8:9.
3. Draw an inch line and divide it into tenths; into twelfths.
4. Draw two lines, AB, 2 in. long; AC 3 in. long. In AC take point D so that AD is 1 in. long. Join BC and draw DE parallel to BC meeting AB in E. Calculate the length of AE. (Note that $\frac{AC}{AB} = \frac{AD}{AE}$. AE is said to be a fourth proportional to AC, AB, AD.)
5. Use the method of the preceding exercise to find a fourth proportional to three lines of lengths: 2, $1\frac{1}{2}$, 1 inches.
6. Find a fourth proportional to lines: $\frac{1}{2}$, $\frac{3}{4}$, $1\frac{1}{2}$ inches.
7. Draw AB, AC, making an angle of 47° with one another. In either of them take a point P and drop a perpendicular PN on the other. Measure the lengths of the sides of APN, and obtain the numerical values of the following ratios to two decimal places,—

$$\frac{PN}{AP}, \quad \frac{AN}{AP} \quad \text{and} \quad \frac{PN}{AN}.$$

(Most accurate results will be obtained by taking P at some distance from A, and measuring in sixteenths of an inch.)

8. Take P in other line, at different distance from A, make similar construction, measure sides of APN, and again find, to two decimal places, the values of the above ratios for 47° .

9. Calling the side opposite 47° the perpendicular, the side opposite the right angle the hypotenuse, and the remaining side the base, whether it be on the upper or lower line, are the above ratios, *i.e.*,

$$\frac{\text{perp.}}{\text{hyp.}}, \quad \frac{\text{base}}{\text{hyp.}} \quad \text{and} \quad \frac{\text{perp.}}{\text{base}},$$

always the same for 47° , or do they depend on where the point P is taken?

10. With the explanation in the preceding question, find the values of these same ratios

$$\frac{\text{perp.}}{\text{hyp.}}, \quad \frac{\text{base}}{\text{hyp.}} \quad \text{and} \quad \frac{\text{perp.}}{\text{base}},$$

for an angle of 63° , to two decimal places.

11. It is required to find the distance of a point C from an object B on the other side of a chasm. For this purpose a line CA is run at right angles to BC. AC is found to be 278 feet, and the angle to A to be 47° . What is the distance of B from C?

12. In the preceding question, if AC be 344 feet, and the angle at A be 63° , what is the distance of B from C? Find also the length of AB.

13. To find how far a distant object C is from A, a base line AB is measured of 400 ft. and the angles at A and B are found to be 75° and 80° . Then, on paper, a line DE of length 3 in. is drawn, and angles EDF, DEF are constructed of 75° and 80° , respectively,—and FD is measured in inches and fractions of an inch. What, approximately, is the length of CA?

14. If, in the preceding question, AB be 250 feet, and the angles at A and B be 65° and 77° , respectively, by constructing a similar triangle on paper and measuring the sides, determine approximately the distances AC and BC.

15. In triangle ABC, $AC=372$ feet, $A=48^\circ$, $C=90^\circ$. Find approximately the length of BC, having previously found for 48° the ratio $\frac{\text{perp.}}{\text{base}}$.

16. Draw an irregular quadrilateral, and construct another of same shape and with linear dimensions half those of former. Verify equality of corresponding angles, and ratio of sides and of diagonals.

17. Draw an irregular pentagon, and construct another of same shape and with linear dimensions one-third those of former. Verify equality of corresponding angles, and ratio of sides and of diagonals.

18. Make an irregular drawing to represent a lake, and construct a second figure of the same shape (*a*) with dimensions half those of the first figure; (*b*) with dimensions twice those of the first figure.

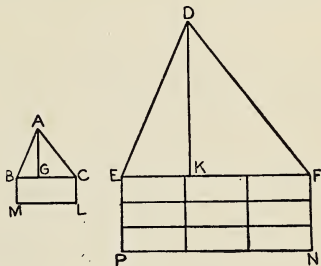
19. A man wishing to ascertain the height of a tower, fixes a rod 11 feet in length vertically in the ground at a distance of 80 feet from the tower. On retiring 10 feet farther from the tower, he sees the top of the rod in a line with the top of the tower. If the observer's eye is $5\frac{1}{2}$ ft. above the ground, find the height of the tower.

20. Construct a triangle with sides 50, 30 and 48 millimetres. Bisect the angle opposite the last side. In what ratio are the segments into which this bisecting line divides this side? Does the same ratio exist elsewhere in the figure?

CHAPTER XX.

Similar Triangles. (Continued.)

1. Let ABC and DEF be similar triangles, having the base EF three times the base BC . The other sides of DEF are therefore three times the corresponding sides of ABC . If DK and AG be the perpendiculars to the bases, the triangles ABG and DEK are equiangular, and therefore, since DE is three times AB , DK is also three times AG .



If rectangles be constructed on the bases equal to the triangles, the heights of these rectangles are half the heights of the triangles (Ch. VIII., 5). Hence FN , which is half of DK , is three times CL , which is half of AG .

So that the rectangle $EFNP$ (which is equal to the triangle DEF) is three times as long and three times as high as the rectangle $BCLM$ (which is equal to the triangle ABC). Hence the rectangle $EFNP$ is nine times the rectangle $BCLM$, and, therefore, the triangle DEF is nine times the triangle ABC .

That is, when

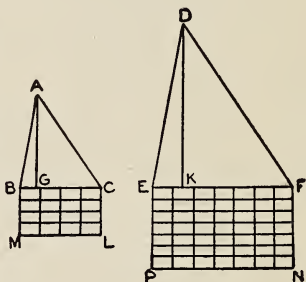
$$\text{side } BC : \text{side } EF = 1:3,$$

$$\text{then, triangle } ABC : \text{triangle } DEF = 1:3^2,$$

the triangles being, of course, similar.

2. Again, let **ABC** and **DEF** be similar triangles, having the base **EF** one and three-quarter times the base **BC**. That is, the base **BC** is to the base **EF** as 4 is to 7, since $1:1\frac{3}{4}=4:7$. Since the triangles are similar, the other sides of **DEF** are $1\frac{3}{4}$ times the corresponding sides of **ABC**. If **AG** and **DK** be the perpendiculars to the bases, the triangles **ABG** and **DEK** are equiangular, and, therefore, since **DE** is $1\frac{3}{4}$ times **AB**, **DK** is also $1\frac{3}{4}$ times **AG**.

If rectangles be constructed on the bases equal to the triangles, the heights of these rectangles are half the heights of the triangles (Ch. VIII., 5). Hence **FN**, which is half of **DK**, is $1\frac{3}{4}$ times **CL**, which is half of **AG**.



So that the rectangle **EFNP** (which is equal to the triangle **DEF**) is $1\frac{3}{4}$ times as long and $1\frac{3}{4}$ times as high as the rectangle **BCLM** (which is equal to the triangle **ABC**). That is, of such parts as **EF** contains 7, **BC** contains 4; and of such parts as **FN** contains 7, **CL** contains 4. Hence of such small areas as the rectangle **EFNP** contains $7^2=49$, the rectangle **BCLM** contains $4^2=16$. And therefore the triangle **ABC** is to the triangle **DEF** as 16 is to 49.

That is, when

$$BC : EF = 1 : 1\frac{3}{4} = 4 : 7,$$

then, triangle **ABC** : triangle **DEF** = $16 : 49 = 4^2 : 7^2$

or $1 : (1\frac{3}{4})^2$.

3. Make figures as in § 1 and § 2 for the following problems :

Two similar triangles, **ABC** and **DEF**, have their corresponding sides **BC** and **EF**, 1 and 2 inches in length respectively ; show that their areas are as 1 to 4, *i.e.*, as 1 to 2^2 .

Two similar triangles, **ABC** and **DEF**, have their corresponding sides **BC** and **EF**, 1 and $1\frac{1}{2}$ inches in length respectively ; show that their areas are as 4 to 9, *i.e.*, as 1 to $(1\frac{1}{2})^2$.

Two similar triangles, **ABC** and **DEF**, have their corresponding sides **BC** and **EF**, 30 and 50 sixteenths of an inch in length respectively ; show that their areas are as 9 to 25, *i.e.*, as $(30)^2$ to $(50)^2$.

(For the three preceding constructions, the method of article 4, which follows, should also be employed.)

The result of our observations in such cases as the preceding may be stated thus :

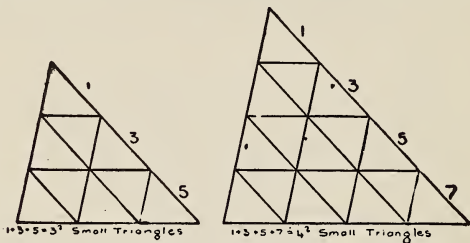
Similar triangles are to one another as the squares of corresponding sides.

Note.—In the preceding examples it will be observed that the lengths of the corresponding sides are supposed *commensurable*, *i.e.*, a unit of length can be found that is contained in both an exact number of times. All lines are not commensurable, though the preceding statement in black-face is true of all similar triangles, whether the corresponding sides be commensurable or not.

4. The following is possibly a more striking way of presenting the preceding proposition :

Let any side, say the base, of a triangle be divided into as many parts as it contains units of length. Through the points of division draw lines parallel to

the sides, and, through the points of intersection of these lines, draw lines parallel to the base. The triangle is thus divided into a number of triangles equal to one another in all respects, and all similar to the original triangle. It will be observed that, considering these triangles in rows, the rows contain 1, 3, 5, 7, . . . triangles, respectively. Hence if the base be 2 units



in length, the large triangle contains $1+3=2^2$ small triangles; if 3 units in length, $1+3+5=3^2$ small triangles; if 4 units in length, $1+3+5+7=4^2$ small triangles; and so on. Thus if there be two similar triangles, the base of one containing 3 units of length, and the base of the other 4 units of length, the number of small triangles in one will be 3^2 , and in the other 4^2 , all such triangles being equal to one another. Hence the areas of the triangles are as 3^2 to 4^2 , *i.e.*, as the squares of the bases.

Summary.

The areas of similar triangles are to one another as the squares on corresponding sides.

Note.—To teachers who may find the book-work of this chapter beyond the scope of their classes, it is suggested that pupils be made familiar at least with the result obtained above.

Pupils may satisfy themselves of its truth by drawing several pairs of similar triangles, and drawing perpendiculars from any vertex upon the opposite side in each. By measuring these perpendiculars and the bases upon which they fall, the area of each triangle may be determined.

$$\text{Area} = \frac{\text{Base} \times \text{Perpendicular}}{2}$$

(*e.g.*) Draw a base 14 sixteenths of an inch long, and on it construct very carefully a triangle whose remaining sides are 13 and 15 sixteenths of an inch. On the base drop a perpendicular from the opposite angle and measure it carefully. (Its length should be 12 sixteenths of an inch.) The area of this triangle is therefore

$$\frac{14}{16} \times \frac{12}{16} \times \frac{1}{2} = \frac{21}{64} \text{ of a square inch.}$$

On a base 28 sixteenths of an inch long, construct a triangle similar to that already drawn, and find its area.

On a base 42 sixteenths of an inch long, construct a triangle similar to those already drawn, and find its area.

It will be noticed that in these three triangles corresponding sides are in the ratio 1:2:3.

Their areas should be in the ratio 1:4:9.

Exercises.

1. Construct two triangles, the sides of one being 2, 3 and 6 quarters of an inch, and of the other 4, 6 and 12 quarters of an inch. Why are these triangles similar? On the base of each construct a rectangle equal in area to the triangle. Divide each side of the first rectangle into two equal parts and each side of the second into four equal parts. By drawing parallel lines as in section 1, show that the areas of the triangles are as $2^2:4^2$.

2. Show in a similar way that the areas of these triangles are as $3^2:6^2$ or as $6^2:12^2$.

3. Draw two straight lines which are to one another as these triangles.

4. Divide a line $3\frac{1}{2}$ in. in length into two segments, such that, when equilateral triangles are described on the segments, one triangle shall be four times the other.

Construct the equilateral triangles, and divide the greater into four triangles, each equal to the smaller.

5. Construct two triangles on bases 3 and 2 inches, angles adjacent to each base 70° and 50° . Divide the triangles into smaller ones, all equal to one another, showing that the areas of the triangles are as $(3)^2$ to $(2)^2$.

6. Draw a line AB of length 1 in., and produce it to C so that AB may be to BC as the areas of the two triangles in the preceding question.

7. Describe an irregular pentagon, and, after the manner of § 5, Ch. XIX., construct another pentagon with linear dimensions half those of former. Divide each pentagon into three triangles by lines drawn from corresponding angles.

How are the sides and angles of corresponding triangles related? Test with bevel and dividers.

How many times is a triangle in the first pentagon greater than the corresponding triangle in the second?

How many times is one pentagon greater than the other?

8. ABC is any triangle, and in AB a point D is taken such that AD is one-quarter of AB. DE is drawn parallel to BC. What fractional part is ADE of the whole triangle? What ratio does ADE bear to the rest of ABC?

9. Construct a right-angled triangle with sides 30, 40 and 50 sixteenths of an inch. On the sides describe equilateral triangles. Divide the triangles into smaller ones, so that the smaller ones may all be equal to one another. What relation do you discover between the area of the triangle on the hypotenuse and the areas of the two other triangles?

10. In the preceding question, instead of equilateral triangles, construct triangles with angles adjacent to the sides of 50° and 80° , so that the three triangles are similar. Again compare areas of smaller triangles with area of greatest.

11. The areas of the provinces of the Dominion being,—P.E.I., 2000; N. S., 20600; N. B., 28200; M., 73956; O., 222000; Q., 347350; B. C., 383300 square miles; and the square roots of these numbers being 45, 144, 168, 272, 471, 589, 619, or approximately as 5, 14, 17, 27, 47, 59, 62; construct seven equilateral triangles, all with same vertex, whose areas shall represent proportionately the areas of the provinces.

12. Draw also seven parallel lines, near one another, and all terminated at one end by the same straight line to which they are

perpendicular, so that these lines may approximately represent the areas of the provinces.

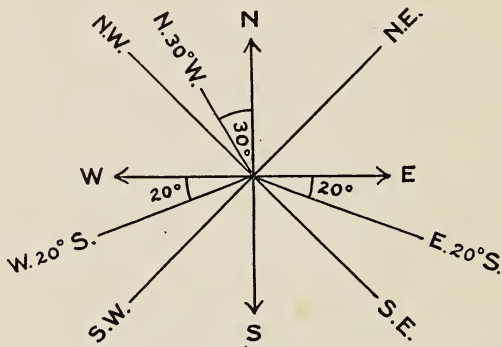
13. Given the following areas,—England, 50867 ; Ireland, 32583 ; Scotland, 29785 ; Wales, 7442, construct four squares, with one angle in common, which shall represent proportionately and approximately the areas of these countries.

14. Draw also four parallel lines, as in 12, which shall represent approximately the areas of the countries of the United Kingdom.

APPENDIX.

Before attempting to work the accompanying set of exercises, the student is advised to read over carefully the following explanation :—

1. In the accompanying figure are given the principal points of the compass. It will be noticed that the line of direction which bisects the angle between North and



West is called North-West, and the terms North-East, South-East, South-West have similar meanings. The angle between North and West is, of course, 90° , and therefore the angle between North-West and North, or North-West and West, is 45° .

When we say that an object when viewed from a certain point lies N. 30° W., we mean that a line drawn

from the point to the object would make 30° on the West side with a line drawn from the point due North. Give similar meanings to E. 20° S., W. 20° S., etc.

2. In Fig. 1, **O** represents an object viewed from a point **P** below it. **OP** is the line in which **P** is seen from **O**, and **PQ** is a horizontal line drawn through **P**. Here **OPQ** is said to be the **angle of elevation** of **O** with respect to **P**, because the object is **above** the observer's eye.

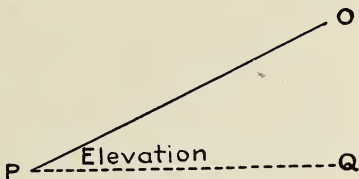


Fig. 1.

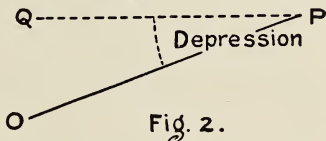


Fig. 2.

In Fig. 2, **O**, the position of the object, is **below** **P**, the position of the observer's eye, and here **QPO** is called the **angle of depression**.

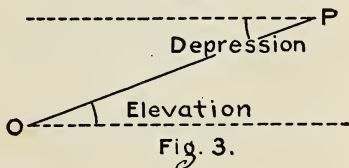


Fig. 3.

From Fig. 3 it will be seen that the angle of elevation of an object **P** viewed from **O** is equal to the angle of depression of an object **O** viewed from **P**.

Note.—The following examples are to be worked by construction and measurement of figures drawn to suitable scale.

Miscellaneous Practical Exercises.

1. If a ship sails 16 miles due North and then 9.5 miles due East, draw a plan showing the course of the ship, and determine by measurement on this plan the distance of the ship from the starting point. Verify by calculation. (Take $\frac{1}{4}$ inch to represent 1 mile.)

2. One ship is 15 miles N.E. and another 25 miles N.W. of a light-house. Taking 1 inch = 5 miles, draw a plan showing the relative positions of the two ships and the light-house. From your plan determine by measurement how far the ships are apart. Verify by calculation.

3. A surveyor starting from a point A moves East to B, a distance of 20 chains; he then goes North to C and finds BC to be 40 chains. Draw a plan on the scale 10 chains = 1 inch, and by measuring your plan find the distance from A to C. Verify by calculation.

4. A man walks North 100 yards, East 200 yards, West 300 yards, South 400 yards. Choosing a suitable scale, draw a plan of his course. Determine by measuring on your plan how far he is from his starting point.

5. Two trains set out from the same station on tracks making an angle of 60° with each other. One train travels at the rate of 30 miles an hour and the other at twice that rate. How far will the trains be apart in 3 hours?

6. Two ships set sail at the same time from the same port, one sailing N. 15° E. and the other W. 15° N. The first ship sails at the rate of 10 miles an hour and the second at 8 miles an hour. Draw a diagram and determine from it how far the ships will be apart in one hour and a half.

7. At a certain point on the road from A to B, the road divides into two branches, one leading to a lake and the other to B, the branches making an angle of 30° with each other. A man walking from A to B at the rate of 4 miles an hour takes the lake road by mistake, and arrives at the lake $\frac{1}{2}$ hour afterwards. Discovering his mistake, he now sets off for the road to B in a direction at right angles to it, his rate now being only 3 miles per hour. Draw a plan on the scale 1 inch = 1 mile, and determine by measurement and calculation how much time he has lost by his mistake.

8. At a point 200 feet from a tower and on a level with its base, the angle of elevation of its summit is found to be 60° . Draw a figure on a scale of 1 inch = 100 feet, and from it determine by measurement the approximate height of the tower.

9. A man walks East 200 yards, changes his course to North 20° West and walks 300 yards. From there he returns to his starting-point. Draw a figure and from it determine the entire length of his walk, and also the area of the triangle formed by his course.

10. The shadow of a man 6 feet high cast by a light 18 feet distant is 9 feet long. How high is the light from the ground?

11. Two forts defend the mouth of a straight river 800 yards wide. Each fort is equipped with guns having an effective range of 600 yards. Draw a figure on a scale of 1 inch = 200 yards showing the portion of the river which is exposed to fire from both forts.

12. From the top of a rock 250 feet above the sea-level, the angle of depression of a ship's hull is found to be 30° . How far is the ship distant?

13. From the top of a cliff 150 feet high, the angles of depression of two boats which are due south of the observer are 15° and 75° . Find their distance apart. (Scale, 1 inch = 100 feet.)

14. From the top of a hill the angles of depression of two consecutive milestones on a level road running due south of the observer are 45° and 22° respectively. What is the height of the hill in yards? (1 mile = 1760 yards. Take 1000 yards = 1 inch.)

15. A kite held by a rope 200 yards long has drifted with the wind until the rope makes an angle of 54° with the ground. How high is the kite?

16. From the roof of a house 30 feet high the angle of elevation of the top of a monument is 42° , and the angle of depression of its foot is 17° . Find its height.

17. At 3 p.m. a ship lies S. 34° W. of a light-house and 6 miles distant. At 4 p.m. the ship lies S. 56° E. of the light-house and 8 miles distant. How fast is the ship sailing?

18. The boundaries of a triangular field are found to be 1 chain 90 links, 2 chains 31 links, and 3 chains 77 links. Make a plan (scale, one chain to one inch), and in the plan draw and measure the

length of one altitude of the triangle. Hence find approximately the area of the field in acres.

19. The survey of a triangular field ABC shows that two boundaries AB and AC measure 6 chains and 2 chains 40 links respectively, and that the angle $BAC=75^\circ$. Draw a plan (scale, 2 chains to an inch). Measure the perpendicular from A on BC, and hence calculate the area of the triangular field.

20. A four-sided field ABCD has $BC=13$ chains, $BA=13$ chains, angle $ABC=60^\circ$, $CD=12$ chains, $AD=5$ chains. Draw a plan (scale, 4 chains to an inch). Reduce the resulting figure to an equal triangle, and by measuring its base and altitude determine the area of the field.

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